

THE MATHEMATICAL GAZETTE

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ON THE DEFINITIONS OF ELLIPTIC FUNCTIONS.

By C. A. B. SMITH.

IN practice it is generally as inverse functions of elliptic integrals that elliptic functions are needed; and so a simple demonstration of this important property of the functions is to be desired. The important step in such a demonstration is a proof that the inverse of an elliptic integral is a one-valued function, which for every value of the variable is regular or has at worst a pole—that is, is meromorphic over the complete plane (save possibly at ∞).

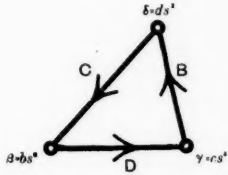
In most books the proof of this property is complex and roundabout. Goursat * gives a very quick and simple proof, based however on the addition theorem, which theorem is a very special property of the functions, and the reasons for it only become clear with some knowledge of the theory. Here is another proof which it is hoped will be of interest: it is longer than Goursat's but it seems more natural.

This new proof is made shorter by the use of the symmetry properties of the elliptic functions. To see how these come about, let us take any three complex numbers, B, C, D , all different from 0, such that

$$B + C + D = 0.$$

The three numbers B, C, D , may be pictured as forming a triangle $\beta\gamma\delta$ in the complex plane (Fig. 1) so that

$$\gamma = \beta + D, \dots \text{and so on.} \dots\dots\dots(1)$$



* Goursat, *Cours d'analyse mathématique*, vol. ii. 3rd ed. (1918), p. 536 (chap. xxi); E. H. Neville, *Jacobian elliptic functions*, pp. 137-139.

(It will only be necessary to put down one equation, keeping in mind that there are two others, $\delta = \gamma + B$, $\beta = \delta + C$, given by the symmetry of the figure.) Now let us take β , γ , δ , to be functions of some variable x , in writing

$$\beta = (bs\ x)^2, \dots; \dots\dots\dots(2)$$

we will then clearly have

$$d(bs^2)/dx = d(cs^2)/dx = d(ds^2)/dx, \dots\dots\dots(3)$$

and so if we put each of these equal to $-2\ bs \cdot cs \cdot ds$, we still have symmetry between bs , cs , and ds . One solution of this equation

$$d(bs^2)/dx = 2\ bs \cdot d(bs)/dx = -2\ bs \cdot cs \cdot ds$$

is for bs , cs , ds to be constants: this will be ruled out as uninteresting. If not, then by (1) none of bs , cs , and ds is zero, and so on division by $2\ bs$ we get

$$d(bs)/dx = -cs \cdot ds \dots\dots\dots(4)$$

or by (1)

$$(dbs/dx)^2 = cs^2 ds^2 = (bs^2 + D)(bs^2 - C) \dots\dots\dots(5)$$

with, naturally, differential equations like this for $cs\ x$, $ds\ x$.

The solution of equation (5) is

$$x = \int_{bs\ x}^{\text{const.}} \frac{dy}{\sqrt{(y^2 + D)(y^2 - C)}},$$

so that $bs\ x$ is the inverse of the elliptic integral on the right-hand side. (In this way we see that any elliptic integral of this form has as its inverse function $bs\ x$, with the right values of the constants C and D . As a special example we may put $D = -1$, $C = k^2$, $B = k'^2$. The functions bs , cs , ds then become the common Jacobian functions $ns = 1/sn$, $cs = cn/sn$, $ds = dn/sn$, so that these functions are covered by this result.)

Our line of attack on the chief result is through

Theorem 2. *There is a number $\rho > 0$, dependent on B, C, D , but not on x , such that if $(x = x_0, bs\ x = b_0)$ is any point on the solution of the differential equation (5), then $bs\ x$ is meromorphic in the circle $|x - x_0| \leq \rho$.*

From this it will not be hard to see that $bs\ x$ is meromorphic in the complete x -plane save possibly for $x = \infty$.

For the proof of Theorem 2 a lemma will be needed:

Theorem 1. *Let $(x = x_0, y = y_0)$ be any point on the solution of the equation*

$$dy/dx = f(y) \dots\dots\dots(6)$$

and let $f(y)$ be regular in the circle $|y - y_0| \leq r$, and let $|f(y)| \leq M$ in this circle.

Then y is a regular function of x in the circle $|x - x_0| < R = r/2M$.

We see this by a comparison of the power series for the function $f(y)$, let us say

$$f(y) = f_0 + f_1(y - y_0) + f_2(y - y_0)^2 + \dots \dots\dots(7)$$

with the power series for the function

$$g(y) = \frac{M}{1 - (y - y_0)/r} = M + Mr^{-1}(y - y_0) + Mr^{-2}(y - y_0)^2 + \dots \dots\dots(8)$$

Because, by Cauchy's integral,

$$f_m = \frac{1}{2\pi i} \oint_{|y - y_0| = r} f(y) \cdot (y - y_0)^{-m-1} dy,$$

$$|f_m| \leq Mr^{-m}, \dots\dots\dots(9)$$

we have

so that every coefficient in the series (7) has modulus less than the coefficient of the same power of $(y - y_0)$ in (8). We now get the solution of (6) as a power series. Let us put $(y - y_0) = \sum_1^{\infty} h_s (x - x_0)^s$ into the equation (6); then we get relations giving us the values of h_1, h_2, h_3, \dots in turn; in fact each h_s is a polynomial in the h_r and f_r for $r < s$, with all its coefficients ≥ 0 .

In the same way, putting $(y - y_0) = \sum_1^{\infty} H_s (x - x_0)^s$ in the equation $dy/dx = g(y)$, we get relations for the H_s . From these relations and (9) we see that $|h_s| \leq H_s$, so that the radius of convergence of the solution of $dy/dx = f(y)$ is not less than the radius of convergence of the solution of

$$dy/dx = g(y) = M/[1 - (y - y_0)r^{-1}] \dots \dots \dots (10)$$

But the solution of (10) is

$$y - y_0 = r \pm [r^2 - 2rM(x - x_0)]^{\frac{1}{2}},$$

the power series of which has the radius of convergence $r/2M$. And so the solution of (6) has a power series with this radius, and so is regular in $|x - x_0| < r/2M$.

Now take equation (5). What singularities (if any) there are in the solution of this equation will come where the right-hand side has a singularity, that is, where $bs^2 + D = 0$ (or $cs = 0$), $bs^2 - C = 0$ (or $ds = 0$), and $bs = \infty$. These values of bs will be named the *S-points* of bs .

It will be natural to do the investigation of (5) in parts, one for the values of bs not near any S-point, and one for the values of bs near each S-point in turn. To this end let $a =$ the least of $|B|/3, |C|/3, |D|/3$, and $A =$ the greatest of $3|B|, 3|C|, 3|D|$, and let a division be made of the values of bs into four sorts,

- P_{2b} , those values for which $|cs^2| < a$
- P_{3b} , those values for which $|ds^2| < a$
- P_{4b} , those values for which $|bs^2| > A$
- P_{1b} , the rest of the possible values.

(In the same way, the function cs will have its S-points, those points at which $ds = 0, bs = 0, cs = \infty$, and its divisions $P_{2c}, P_{3c}, P_{4c}, P_{1c}$.)

(i) First let us take the values of bs in P_{1b} , those values not near any S-point. There will be a least distance $2r > 0$ between these values and the S-points, so that if $bs x_0 = b_0$ is a point in P_{1b} , then $\sqrt{[(bs^2 + D)(bs^2 - C)]}$ is a regular function of bs in $|bs - b_0| \leq r$.

In addition there will be a greatest value M of $\sqrt{[(bs^2 + D)(bs^2 - C)]}$ over all points of P_{1b} and of distance $\leq r$ from P_{1b} . By Theorem 1 we now see that bs is regular in $|x - x_0| < r/2M = R_1$ (say) > 0 .

By symmetry, if $cs x_0$ is in P_{1c} , cs is regular in $|x - x_0| < R_2$ (say) > 0 .

(ii) Now take x_0 to be in P_{2b} so that $|cs^2| < a$. By Fig. 1 we see that $cs x_0$ is not near any of its S-points, so that (by (i)) it is regular in $|x - x_0| < R_2$. From this $bs x = \sqrt{(cs^2 - D)}$ is regular in $|x - x_0| < R_2$.

(iii) In the same way if $bs x_0$ is in P_{3b} , $bs x$ is regular in $|x - x_0| < R_3$ (say) > 0 .

(iv) If $bs x_0$ is in P_{4b} , and so is large, it is natural to make use of the variable $1/bs$. In fact, putting $u = \sqrt{(-CD)/bs}$, we get from (5) the equation for u ,

$$(du/dx)^2 = (u^2 + D)(u^2 - C),$$

which is of the same form as (5). But because $|bs|^2 > A$, $|u^2| < a$, so that u is in P_{1b} , and so by (i) u is regular in the circle $|x - x_0| < R_1$. From this, bs is meromorphic in this circle.

In short, if ρ is the least of R_1, R_2, R_3 , and if $(x_0, bs x_0)$ is any point on the solution of equation (5), then $bs x$ is meromorphic in the circle $|x - x_0| < \rho$.

Theorem 3. bs is meromorphic in the complete x -plane (but for $x = \infty$).

For by Theorem 2, if $bs x$ is meromorphic in $|x - x_0| \leq n\rho/3$, it is meromorphic in $|x - x_0| \leq (n+1)\rho/3$. And it is meromorphic in $|x - x_0| \leq \rho/3$. So by induction it is meromorphic in any such circle, however large.

In this proof we have made use of three functions, bs, cs, ds , between which there is cyclic symmetry. It may be noted that the use of these functions is a help, not only here, but in addition in the development of the rest of the theory, such as the existence of periods, addition formulae, and so on.*

As an example of the use of these functions, let us see how the theory may be based on Liouville's Theorem, in place of using the properties of the differential equation (5). (This line of attack is one commonly used for the Weierstrassian functions, but, as we shall see, it may equally well be used for the Jacobian.)

The functions bs, cs, ds , are not completely fixed by the differential equation (5): if we make an addition of any constant to x , the equation will still be true. We will therefore make the further condition that bs, cs, ds are to have simple poles of residue 1 at 0. These functions are then completely fixed. It may now be seen, with a little trouble, that we may take a root ω_b of the equation

$$bs\omega_b = 0, \dots \dots \dots (11)$$

and in the same way, numbers ω_c, ω_d , such that †

$$\omega_b + \omega_c + \omega_d = 0, \dots \dots \dots (12)$$

and none of $\omega_b/\omega_c, \omega_c/\omega_d, \omega_d/\omega_b$ is real. $\dots \dots \dots (13)$

and $bs x$ has periods $2\omega_b, 4\omega_c, 4\omega_d$, and simple poles of residue +1 at points $\equiv 0$ and $2\omega_b$, and of residue -1 at points $\equiv 2\omega_c$ and $2\omega_d \pmod{4\omega_b, 4\omega_c, 4\omega_d}$, and is regular at all other points.

But another function with these properties is the function

$$Bs x = \sum_{m, n = -\infty}^{\infty} \left[\frac{(x - \Omega)^{-1} + (x - 2\omega_b - \Omega)^{-1}}{-(x + 2\omega_c - \Omega)^{-1} - (x + 2\omega_d - \Omega)^{-1}} \right], \dots (14)$$

$$\Omega = 4m\omega_c + 4n\omega_d$$

where the series on the right-hand side is made convergent by grouping the terms together in fours. We see from this definition that

$$Bs \omega_b = Bs(-\omega_b) = 0, \dots \dots \dots (15)$$

and that the function Bs has the periods $4\omega_c, 4\omega_d$. $\dots \dots \dots (16)$
(In the same way we shall have functions Cs, Ds , with like properties.)

We now make use of Liouville's Theorem, in the form that if two one-valued functions, f, g , have two independent periods in common, and if their difference $f - g$ is everywhere regular, then $f = g + \text{const.}$ $\dots \dots \dots (17)$

* We do not go into this in detail here, as in a letter Prof. Neville has said that there is a full discussion in his new book, *Jacobian Elliptic Functions*. See in addition M. M. U. Wilkinson, "Elliptic and Allied Functions", *Proc. 5th Inter. Cong. Math.* (1912), vol. i, p. 407.

† For detailed proofs see E. H. Neville, *Jacobian Elliptic Functions*.

From this, $Bs - bs$ is clearly constant, and putting $x = \omega_b$, we see that it is 0, that is, $Bs = bs$. So, starting from the definition (5) of the function bs , which is based on the numbers B, C, D , we may get the expansion (14).

On the other hand, if we take any three numbers $\omega_b, \omega_c, \omega_d$, with the properties (12) and (13), the expansion (14) will give us a function $Bs x$ having the properties (15) and (16), and simple poles of residue 1 at 0 and $2\omega_b$, and residue -1 at $2\omega_c, 2\omega_d$.

By (17) we then have these relations

$$Bs x + Bs(-x) = \text{const.} = 0 \text{ (putting } x = \omega_b) \dots\dots\dots (18)$$

$$Bs(x + 2\omega_b) - Bs x = \text{const.} = 0 \text{ (putting } x = -\omega_b) \dots\dots\dots (19)$$

$$Bs(x + 2\omega_c) + Bs x = \text{const.} = 0 \text{ (putting } x = -\omega_c, \text{ and using (18))} \dots\dots (20)$$

$$Bs(x + 2\omega_d) + Bs x = 0 \text{ (in the same way).} \dots\dots\dots (21)$$

Putting $(d Bs/dx) = Bs' x$, we have from (18),

$$Bs' x = Bs'(-x), \text{ and from (20), } Bs' x = -Bs'(x + 2\omega_c).$$

$$\text{So putting } x = -\omega_c, \text{ we see that } Bs' \omega_c = 0. \dots\dots\dots (22)$$

Now from (18) we see that the expansion of Bs near 0 is of the form $x^{-1} + O(x)$, and the same is true of Cs, Ds . From this, the function $Cs^2 - Bs^2$ is regular at 0. From relations like (19), (20) we see that it is regular at $2\omega_b, 2\omega_c, 2\omega_d$, and so for all x , so that it is, by (17), a constant. If then we let $Cs\omega_b = \sqrt{D}$, $\dots\dots\dots (23)$

$$\text{we have } (Cs x)^2 - (Bs x)^2 = D \dots\dots\dots (24)$$

In the same way we see that

$$Bs' x + Cs x. Ds x = \text{const.} = 0 \text{ (letting } x = \omega_c, \text{ and using (22))} \dots\dots\dots (25)$$

and putting in this equation the values of Cs and Ds given by (23), and squaring, we get $(Bs')^2 = (Bs^2 + D)(Bs^2 - C)$.

But this is the differential equation (5). So the function Bs got from the expansion (14) is equal to the function bs , got from the equation (5), with constants B, C, D , given by (23). Now we have seen that every function bs is of the form Bs , so that we may say that the differential equation (5), together with the condition that there is to be a pole of residue 1 at 0, is completely equivalent to the expansion (14).
C. A. B. S.

THE GENERAL MEETING

ANY doubts which the Executive Committee may have felt in summoning a general meeting during war-time were made to seem foolish by the large and enthusiastic audiences which gathered at King's College, London, on April 12 and 13. A full report of the proceedings, with some account of the papers and discussions, will appear in a later issue of the *Gazette*. In a brief notice such as this, it is enough to note that the new President is Mr. C. O. Tuckey, and that the two discussions, "Possible Changes in the Mathematical Syllabus for the School Certificate Examination", and "The Mathematical Course for Sixth Form Scientists", drew keen, critical and argumentative audiences.

The hospitality of King's College was as generous and the efficiency of its arrangements as admirable as in pre-war days.

The organisation of a meeting in April, 1945 is being undertaken, and members will in due course receive the usual circular asking for suggestions for topics for papers and discussions; a prompt reply will materially assist the work of the Programme Committee.

CHESS IN THREE AND FOUR DIMENSIONS.

BY N. M. GIBBINS.

1. In Note 1568 (V for Victory) very brief reference was made to Generalised Chess, that is, an expansion of the ordinary game as played on the 8×8 board. In that note attention was called to the existence of the 5-leaper, whose moves are from $(0, 0)$ to $(5, 0)$; $(0, 5)$; $(3, 4)$; $(4, 3)$. Here it is proposed to give some account of chess in three and four dimensions. In anticipation it must be pointed out that fractions do not exist in chess geometry—a special kind of lattice geometry; but that lengths such as $\sqrt{(a^2 + b^2)}$ and $\sqrt{(a^2 + b^2 + c^2)}$, which are not integral, but where a, b, c are, do exist, since they can be constructed from integral lattices.

2. The following is an abstract of a description by Mr. T. R. Dawson of "Space Chess", or chess in three dimensions (*Fairy Chess Review*, 1943, Vol. V, p. 40). "The normal game is played in a $5 \times 5 \times 5$ cube of 125 cells. The Rook moves through cell faces, the Bishop through cell edges, and the Unicorn [a new piece] through cell corners. The Queen combines the powers of the foregoing pieces and the King moves like a Queen but only in single steps. The Knight moves in all planes like an ordinary Knight. Pawns move rookwise single steps and capture bishopwise single steps, and there is no double-step move." In order to exhibit these moves on a plane we imagine the cube to be made up of five 5×5 boards placed one above the other at unit intervals and label them A, B, C, D, E , starting from the bottom (Fig. 1). The files of each board are labelled a, b, c, d, e ; and the ranks 1, 2, 3, 4, 5. The top boards are then lifted and placed in order in the plane of A . All these features together with the game array are shown in Fig. 1, where inverted Knights denote Unicorns. It may be added that the White Pawns move towards their promotion ranks at $Ea5$ to $Ee5$; and the Black Pawns towards $Aa1$ to $Ae1$.

The Bishop in plane chess divides the lattice into a two-colour system, the usual white and black squares; and this system can also separate the cells in space-chess. But also the Unicorn can divide the cube into a four-colour system, a Unicorn in a cell of any one colour being unable to visit cells of any other colour. Such a colour scheme is indicated by the numbers 1, 2, 3, 4 placed in vacant squares in Fig. 1, each colour reaching up to the board above it, and for board E a unit distance up.

Closed Knight's tours play a considerable part in the literature of ordinary chess, a Knight placed on any square having to visit every one of the other 63 squares once and once only. The smallest lattice in which this can be done in space-chess is shown in Fig. 2, a hitherto unpublished example by E. Huber-Stockar, Geneva. The Knight starts at $Bc1$, and goes for its first move to $Db1$. A more ambitious example is given in Fig. 3, by F. Maack, *Mitt. über Raumchack*, 1909, No. 2, p. 31. It is based on a sequence of rhomboid cycles 1-8, 9-16, 17-24, etc.

3. It will be convenient to take as origin O the centre of $Aa1$, as axis of x a line parallel to ranks, as axis of y a line parallel to files, and as axis of z a line upwards (A to E).

Since $3^2 = 1^2 + 2^2 + 2^2$, a "sphere" 3-leaper placed at O commands $(3, 0, 0)$; $(0, 3, 0)$; $(0, 0, 3)$; $(1, 2, 2)$; $(2, 1, 2)$; $(2, 2, 1)$. This is the only sphere leaper that can exist in the normal cube, and if placed in the centre cell it commands 24 cells. Composers of problems in space-chess may perhaps find a use for it.

We also have $9^2 = 8^2 + 4^2 + 1^2 = 7^2 + 4^2 + 4^2 = 6^2 + 6^2 + 3^2$

while

$$7 + 4 + 4 = 6 + 6 + 3 = 15 = 5 + 5 + 5.$$

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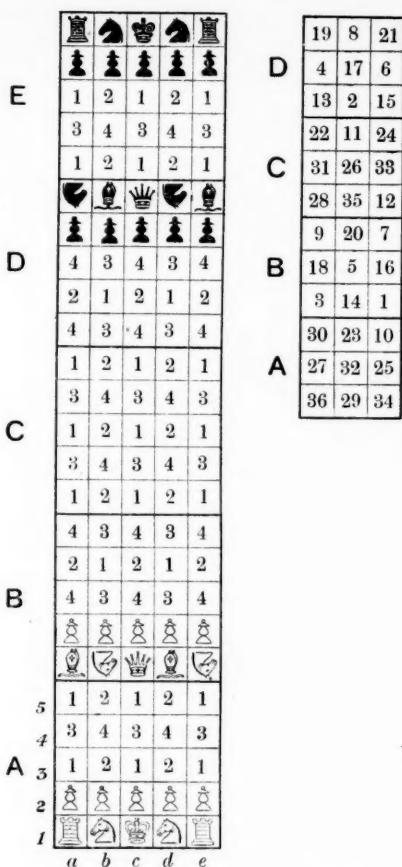


FIG. 1.

D	19	8	21
	4	17	6
	13	2	15
	22	11	24
C	31	26	33
	28	35	12
	9	20	7
B	18	5	16
	3	14	1
	30	23	10
A	27	32	25
	36	29	34

FIG. 2.

Hence the three points (7, 4, 4) and the three points (6, 6, 3) are all equidistant $\sqrt{(2^2 + 1^2 + 1^2)}$ from (5, 5, 5) in their plane. The six points are vertices of a regular hexagon, their order being (7, 4, 4); (6, 6, 3); (4, 7, 4); (3, 6, 6); (4, 4, 7); (6, 3, 6). Joining up all the points, the model looks in one aspect like a partially closed umbrella (without the protection); and in another aspect like the hand-brake used by ski-ers.

Again, $15^2 = 14^2 + 5^2 + 2^2 = 12^2 + 9^2 = 11^2 + 10^2 + 2^2 = 10^2 + 10^2 + 5^2$

so that the two sets of points (14, 5, 2) and (12, 9, 0) are all distant $\sqrt{(7^2 + 5^2 + 2^2)}$ from the coplanar point (7, 7, 7). The twelve points interlock

to form two regular hexagons, the orders of the vertices being respectively (14, 5, 2); (9, 12, 0); (2, 14, 5); (0, 9, 12); (5, 2, 14); (12, 0, 9); (12, 9, 0); (5, 14, 2); (0, 12, 9); (2, 5, 14); (9, 0, 12); (14, 2, 5). The vertices are alternately $\sqrt{(4^2 + 2^2 + 2^2)}$ and $\sqrt{(3^2 + 3^2)}$ apart, leading to the odd result that

$$\arccos(22/26) + \arccos(23/26) = \arccos \frac{1}{2},$$

all the angles being acute.

The radius of the hexagon can never be an integer, since

$$\Sigma(n_1 - \frac{1}{3}\Sigma n)^2 = \frac{1}{3}\{(n_2 - n_3)^2 + (n_3 - n_1)^2 + (n_1 - n_2)^2\}.$$

D	35	24	61	10
	52	7	46	25
	23	36	9	62
	8	51	26	45
C	64	11	34	21
	47	28	49	6
	12	63	22	33
	27	48	5	50
B	53	2	43	32
	38	17	60	15
	1	54	31	44
	18	37	16	59
A	42	29	56	3
	57	14	39	20
	30	41	4	55
	13	58	19	40

FIG. 3.

When we project the cells of the cube n^3 in the manner of Fig. 1, the point (a, b, c) in the cube becomes $(a, b + cn)$ in the plane, while the line

$$(x - a)/\lambda = (y - b)/\mu = (z - c)/\nu$$

becomes

$$(x - a)/\lambda = \{y - (b + cn)\}/(\mu + \nu n);$$

e.g., the Unicorn line $x = y = z$, when $n = 5$, becomes the line $y = 6x$ in Fig. 1. The point $(7, 4, 4)$ becomes $(7, 4 + 4n)$, etc., and we require the projection of the "circle" on which six points of the 9-leaper lie. Eliminating θ and ϕ between the equations

$$x = 9 \sin \theta \cos \phi, \quad y = 9 (\sin \theta \sin \phi + n \cos \theta),$$

$$9 (\sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta) = 15,$$

we eventually obtain

$$(n^2 - n + 1) \xi^2 + (n + 1) \xi \eta + \eta^2 = 3(n - 1)^2,$$

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where $\xi \equiv x - 5$ and $\eta \equiv y - 5 - 5n$. This is the equation of an "ellipse" and is satisfied by

$$\xi = 7 - 5 = 2, \quad \eta = 4(n+1) - 5(n+1) = -(n+1), \text{ etc.}$$

4. We may express the motion of the normal "riders" in generalised chess by means of equations as follows:

<i>Dimensions.</i>	<i>Piece.</i>	<i>Parallel path.</i>
2	Rook	$x = 0, y = 0$
	Bishop	$x = \pm y$
3	Rook	$x/1 = y/0 = z/0, \text{ etc.}$
	Bishop	$x/0 = y/1 = z/\pm 1, \text{ etc.}$
	Unicorn	$x = \pm y = \pm z.$
4	Rook	$x/1 = y/0 = z/0 = t/0, \text{ etc.}$
	Bishop	$x/1 = y/\pm 1 = z/0 = t/0, \text{ etc.}$
	Unicorn	$x/1 = y/\pm 1 = z/\pm 1 = t/0, \text{ etc.}$
	Balloon	$x = \pm y = \pm z = \pm t.$

The Balloon is a (1, 1, 1, 1) rider, and the diagonal of a unit hyper-cell is 2. Hence in its aspect as a single-jump leaper it can go from *O* to four hyper-cells like (2, 0, 0, 0). This move has its analogue in plane chess. The (2, 0) or (0, 2) leaper is called a Dabbaba (Persian for "war-chariot", literally "hovel on wheels"). It was first introduced (*circa* A.D. 800) in Tamerlane's "Great Chess". Tamerlane, a Tartar, did not invent the game which bears his name but "was very fond of playing it". Since all the new pieces in this game have Persian names, Great Chess may be presumed to be a Persian development of Muslim Chess.

5. Just as we represented the cube in space chess by a set of boards as in Fig. 1, so we can realise hyper-space chess by a row of models I, II, III, etc., each like Fig. 1 and placed side by side. In space chess a Unicorn at *Aa1* (say) cannot reach *Aa2*, *Ab1* or *Ba1*, the adjacent squares at unit rook distance. But in hyper-space chess he can; for example *IAa1* to *IBb2*, *IIAb1*, *IAa2*; or (0, 0, 0, 0) to (0, 1, 1, 1); (1, 0, 1, 0); (0, 0, 0, 1). This was pointed out by Mr. Dawson in *Chess Amateur*, 1926, Vol. XXI, p. 92. Furthermore, in

B	11	2	14	7
	4	13	9	16
A	6	15	3	10
	1	8	12	5
	I		II	

FIG. 4.

Fig. 4, he shows a closed Unicorn tour in the 2^4 lattice, and points out that this tour is equivalent to stating all the permutations of four zeros and four ones, where we can repeat either as often as we need. The steps are: (0, 0, 0, 0); (0, 1, 1, 1); (1, 0, 0, 1); (0, 1, 0, 0); (1, 0, 1, 0); (0, 0, 0, 1); (1, 1, 1, 1); (0, 0, 1, 0); (1, 1, 0, 0); (1, 0, 1, 1); (0, 1, 0, 1); (1, 0, 0, 0); (0, 1, 1, 0); (1, 1, 0, 1); (0, 0, 1, 1); (1, 1, 1, 0); (0, 0, 0, 0).

6. The number of ways in which an integer can be expressed as the sum of four squares increases very greatly as the number increases (*Gazette*, XVIII, 10, 11) and therefore some kind of limitation must be imposed on "hyper-sphere" leapers. It was pointed out in § 3 that the radius of the

"circle" on which certain points, commanded by a sphere leaper, lie can never be integral, but in hyper-space the radius of the corresponding "sphere" can be integral.

If n_1, n_2, n_3, n_4 are any positive integers, and $n_1 + n_2 + n_3 + n_4 = 4s$, then

$$\begin{aligned}\Sigma(n_1 - s)^2 &= \Sigma n^2 - 8s^2 + 4s^2 \\ &= \Sigma n^2 - 4s^2 \\ &= (n_1^2 + n_2^2 + n_3^2 + n_4^2) - \frac{1}{4}(n_1 + n_2 + n_3 + n_4)^2 \\ &\equiv \frac{1}{4}(n_1 + n_2 - n_3 - n_4)^2 + \frac{1}{4}(n_1 - n_2 + n_3 - n_4)^2 + \frac{1}{4}(n_1 - n_2 - n_3 + n_4)^2 \\ &= 4m^2, \text{ say.}\end{aligned}$$

Then $\Sigma n^2 = 4(s^2 + m^2)$, and we can choose s so that this is a perfect square, provided that m is of the form $2^p(2\lambda + 1)$. Let

$$n_1 + n_2 - n_3 - n_4 = 4a, \quad n_1 - n_2 + n_3 - n_4 = 4b, \quad n_1 - n_2 - n_3 + n_4 = 4c.$$

Then $n_1 = n_1 - 2(b + c)$; $n_2 = n_1 - 2(c + a)$; $n_3 = n_1 - 2(a + b)$;

$$s = n_1 - (a + b + c), \text{ or } n_1 = s + a + b + c, \quad a^2 + b^2 + c^2 = m^2.$$

For example, let $a = 1, b = 2, c = 2$; then $m = 3$. Thus $s^2 + 3^2$ is to be a perfect square; hence $s = 4, n = 10$: and $n_1 = 4 + 1 + 2 + 2 = 9, n_2 = 3, n_3 = 3, n_4 = 1$. Also $a = 3, b = 0, c = 0$ gives $n_1 = 7, n_2 = 7, n_3 = 1, n_4 = 1$. Thus the twelve points (9, 3, 3, 1) and the six points (7, 7, 1, 1) are all distant 10 from 0, and distant 6 from (4, 4, 4, 4). Moreover the points (9, 3, 3, 1); (3, 9, 3, 1); (3, 3, 9, 1); (7, 7, 1, 1); (7, 1, 7, 1); (1, 7, 7, 1) are all in the plane $x + y + z = 15, t = 1$ and are all distant $\sqrt{24}$ from (5, 5, 5, 1). They are also vertices of a regular hexagon, the order of the above points being 1, 4, 2, 6, 3, 5, 1. Similarly for other sets.

When m is prime there is only one value of s for which $s^2 + m^2$ is a perfect square; but when m is a composite number there is more than one value; for example, $9^2 + 12^2 = 15^2, 9^2 + 40^2 = 41^2; 15^2 + 20^2 = 25^2; 15^2 + 36^2 = 39^2; 15^2 + 112^2 = 113^2$. Hence we decide to take only the smallest value. Even so, the results are complicated. If, for example, we base a hyper-sphere leaper on the sphere leaper 9 we have

$$\begin{aligned}9^2 &= 8^2 + 4^2 + 1^2 = 7^2 + 4^2 + 4^2 = 6^2 + 6^2 + 3^2; \quad m = 9, \quad s = 12, \quad n = 30. \\ a = 9, b = 0, c = 0 &\text{ gives } n_1 = 21, \quad n_2 = 21, \quad n_3 = 3, \quad n_4 = 3. \\ a = 8, b = 4, c = 1 &\text{ gives } n_1 = 25, \quad n_2 = 15, \quad n_3 = 7, \quad n_4 = 1. \\ a = 7, b = 4, c = 4 &\text{ gives } n_1 = 27, \quad n_2 = 11, \quad n_3 = 5, \quad n_4 = 5. \\ a = 6, b = 6, c = 3 &\text{ gives } n_1 = 27, \quad n_2 = 9, \quad n_3 = 9, \quad n_4 = 3.\end{aligned}$$

Thus the above four sets of points are all distant 30 from O and distant 18 from (12, 12, 12, 12). Moreover the sets (27, 11, 5, 5) and (27, 9, 9, 3) with permutations of (11, 11, 5) and of (9, 9, 3) are in the plane $x = 27, y + z + t = 21$ and are all distant $\sqrt{24}$ from (27, 7, 7, 7); with similar results obtained by putting 27 in the second, third and fourth place. Each set of six points forms the vertices of a regular hexagon.

When we apply this process to 15^2 we obtain - 1 for one of the coordinates. Hence in order to keep within the positive sixteenth part of the lattice we add unity to each of the coordinates, obtaining finally that the sets of points (36, 36, 6, 6); (42, 28, 10, 4); (42, 24, 18, 0); (44, 20, 18, 2); (46, 16, 16, 6) are all distant 50 from (1, 1, 1, 1) and all distant 30 from (21, 21, 21, 21); while the second and third sets, where we keep 42 in the first place and permute the other numbers, are all in the plane $x = 42, y + z + t = 42$, and are distant $\sqrt{312}$ from (42, 14, 14, 14) with three other similar results. Moreover each set of 12 points interlocks to form two regular hexagons.

N. M. G.

A DOUBLY INFINITE SYSTEM OF CYCLIC QUADRILATERALS.

BY E. A. MAXWELL.

DURING the last year or two I have set in various places, from points of view varying with the examination, questions on cyclic quadrilaterals formed by two pairs of tangents to a conic. As this use of the configuration seems to have reached temporary saturation, I pass on an amplified account of the work to a larger, and perhaps more willing, audience.

1. Let
$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

be the equation of a given conic, and $P_1 (\xi_1, \eta_1)$, $P_2 (\xi_2, \eta_2)$ any two points of its plane. Any conic through the four points of intersection of tangents from P_1, P_2 to S has an equation which can be expressed in the form

$$S_1 S - T_1^2 + \lambda (S_2 S - T_2^2) = 0,$$

where $S_i = a\xi_i^2 + 2h\xi_i\eta_i + b\eta_i^2 + 2g\xi_i + 2f\eta_i + c$ ($i = 1, 2$)

and
$$T_i = x(a\xi_i + h\eta_i + g) + y(h\xi_i + b\eta_i + f) + (g\xi_i + f\eta_i + c) \\ = xX_i + yY_i + Z_i, \text{ say.}$$

This conic is a circle provided that

$$aS_1 - X_1^2 + \lambda (aS_2 - X_2^2) = bS_1 - Y_1^2 + \lambda (bS_2 - Y_2^2),$$

and that $hS_1 - X_1Y_1 + \lambda (hS_2 - X_2Y_2) = 0$.

If such a value of λ exists, then

$$\{(a-b)S_1 - X_1^2 + Y_1^2\} / (hS_1 - X_1Y_1) = \{(a-b)S_2 - X_2^2 + Y_2^2\} / (hS_2 - X_2Y_2) \\ = \mu, \text{ say;}$$

and this holds if the points P_1 and P_2 both lie on the conic Σ given by

$$\{(a-b)S - X^2 + Y^2\} / (hS - XY) = \mu,$$

so that we may take the equation of Σ in the form

$$\Sigma \equiv (a-b-\mu h)S - X^2 + \mu XY + Y^2 = 0,$$

where $X \equiv ax + hy + g$, $Y \equiv hx + by + f$ and μ is arbitrary.

Hence the conics S and Σ are so related that, if P_1 and P_2 are any two points whatever of the conic Σ , then the four points of intersection of tangents from P_1 and P_2 to S are concyclic.

It is easy to prove that Σ is a rectangular hyperbola concentric with S .

2. Let us consider some of the more obvious particular cases by taking special forms for S .

2.1. If S is a central conic, we may put $f=g=h=0$, $c=-1$. Then

$$S \equiv ax^2 + by^2 - 1 = 0,$$

$$\Sigma \equiv x^2 - y^2 - \mu xy + (a-b)/ab = 0$$

after some rearrangement.

For an ellipse, we can write $a=1/\alpha^2$, $b=1/\beta^2$, so that $(a-b)/ab = -\alpha^2e^2$, where e is the eccentricity. The conic Σ can then be expressed in the form

$$\Sigma \equiv x^2 - y^2 - \alpha^2e^2 - \mu xy = 0,$$

showing that it is a rectangular hyperbola concentric with S and passing through the foci $(\pm\alpha e, 0)$.

For a hyperbola, we can write $a = 1/\alpha^2$, $b = -1/\beta^2$, so that $(a-b)/ab = -\alpha^2\beta^2$, where e is the eccentricity, and Σ is again a rectangular hyperbola concentric with S and passing through the foci.

2.2. If S is a parabola, we may put $a=c=f=h=0$, $b=1$, $g=-2a$. Then

$$S \equiv y^2 - 4ax = 0,$$

and

$$\Sigma \equiv 2(x-a) - \mu y = 0,$$

so that Σ is an arbitrary line through the focus.

Hence if P_1 and P_2 are any two points whose join passes through the focus of a parabola, then the four points of intersection of tangents from P_1 and P_2 to the parabola are concyclic—a result which can also be established by three lines of pure geometry.

2.3. The case when S is a rectangular hyperbola immediately suggests the question whether the roles of S and Σ can be interchanged.

Putting $a=1$, $b=-1$, $c=-a^2$, $f=g=0$, we have

$$S \equiv x^2 + 2hxy - y^2 - a^2 = 0$$

and

$$\Sigma \equiv x^2 - \mu xy - y^2 - a^2(2 - \mu h)/(1 + h^2) = 0$$

after some rearrangement.

If now we write $-\mu = 2k$, $a^2(2 - \mu h)/(1 + h^2) = b^2$, the equation of Σ becomes

$$\Sigma \equiv x^2 + 2kxy - y^2 - b^2 = 0,$$

and its “ Σ ” is, say, Σ' , where

$$\Sigma' \equiv x^2 - vxy - y^2 - b^2(2 - vk)/(1 + k^2) = 0.$$

For S and Σ to be interchangeable, we must have $\Sigma' \equiv S$, so that

$$-v = 2h, \quad b^2(2 - vk)/(1 + k^2) = a^2.$$

Putting these results together and eliminating μ and v , we have the relation

$$\frac{2 + 2hk}{1 + k^2} \cdot \frac{2 + 2hk}{1 + h^2} \cdot a^2 = a^2,$$

giving

$$4(1 + hk)^2 = (1 + h^2)(1 + k^2).$$

To summarise, writing $a^2 = c^2\sqrt{(1 + h^2)}$ for symmetry, the two conics

$$x^2 + 2hxy - y^2 - c^2\sqrt{(1 + h^2)} = 0,$$

$$x^2 + 2kxy - y^2 - c^2\sqrt{(1 + k^2)} = 0,$$

where $4(1 + hk)^2 = (1 + h^2)(1 + k^2)$, are so related that the pairs of tangents drawn from two arbitrary points of either to the other cut in four concyclic points.

3. To obtain a projective treatment of the problem, we proceed as follows. Let P be any point in the plane of a triangle XYZ (a convenient figure may be drawn by taking P near the middle point of the arc YZ of the circumcircle of the triangle XYZ), and let PX , PY , PZ meet the sides YZ , ZX , XY in points L , M , N respectively. Let a conic Σ be drawn through the points X , Y , Z , P , and let a conic S be drawn to touch the sides XY , XZ , PY , PZ . We prove that a conic can be drawn to pass through M , N and the four points of intersection of tangents drawn to S from any two given points of Σ .

The conics S and Σ can be regarded as related in the same way as a conic S and its director circle Σ , the quadrilateral $XPYZ$ being a rectangle inscribed

in Σ and circumscribed about S . If P_1 and P_2 are two arbitrary points of Σ , the tangents from P_1 and from P_2 to S are at right angles, and their four points of intersection have the configuration of three vertices of a triangle and its orthocentre. Any conic through these four points is therefore a rectangular hyperbola, and since M and N are in perpendicular directions (from the rectangle $XYPZ$) there is a conic of the system whose asymptotes are in these directions—that is, there is a conic through M and N .

Now consider this figure from an alternative point of view, in which M and N are regarded as the circular points at infinity. S is a conic of which, say, Y and Z are the real foci and X and P are the imaginary foci. Σ is a conic through X, Y, Z, P , and we can easily derive the equivalent definition that Σ is a rectangular hyperbola through Y and Z having its centre at L . If P_1 and P_2 are two arbitrary points of Σ , the tangents from P_1 and P_2 to S lie on a conic which also passes through M and N —that is, on a circle. This result is clearly equivalent to the preceding analytical work. E. A. M.

MIDLAND BRANCH.

This Branch held its first General Meeting since the outbreak of hostilities in the University of Birmingham on 27th November, 1943. Mr. Caton, Headmaster of Alcester Grammar School, was in the Chair, and thirty-six members were present.

After the election of officers for the session 1943–1944, Mr. E. V. Smith gave his views on the new alternative School Certificate Syllabus in Geometry and Trigonometry of the Cambridge Local Examinations Syndicate. His talk was followed by tea, after which the meeting was thrown open to discussion. The general ideas of the syllabus were acceptable, and it was felt that there was a strong case for the reduction in the number of theorems (though not such a drastic cut as that given in the syllabus) and for the introduction of more trigonometry and more practical geometry. After certain questions on the Specimen Paper were criticised, it was decided to make the views of the meeting known to the parent Association.

A second meeting was held at the University of Birmingham on 12th April, 1944. Mr. Caton was in the Chair, and there were seventy-six members present.

Mr. A. C. Aldis, of Aldis Brothers, Ltd., consulting and manufacturing opticians, gave a paper entitled "The Mathematician and Humanity". In reviewing the qualifications required by a young mathematician who intended entering on a business career, Mr. Aldis placed among the advantages possessed by a mathematically-trained person (i) an appreciation of what constitutes the necessary and sufficient ability to use terms with exactness, (ii) ability to make a factual record of events. He stressed the fact that unless a young man were something more than a mathematician he would be of little use to business or humanity. Human problems do not allow of cut-and-dried solutions, and often the resolution of a business problem is mainly a matter of compromise between two or more muddled possibilities. In particular, the mathematical laws of probability have only a limited practical application in ordinary life.

Throughout the paper there were flashes of humour which were much appreciated by the audience.

After an interval for tea, Mr. Aldis replied to questions and comments. Dr. Pedoe proposed a vote of thanks to Mr. Aldis, and this was carried with acclamation.

L. E. HARDCASTLE, R. J. FULFORD, *Joint Hon. Secs.*

CORRESPONDENCE.

SCHOOL CERTIFICATE.

To the Editor of the *Mathematical Gazette*.

SIR,—I was very interested by the article "Reflections on the Teaching of Mathematics" by Dr. S. Weikersheimer in the February *Gazette* and in his commendation of our school certificate examination. May I put forward a view on the latter subject, speaking chiefly as a parent?

We have gone a long way in improving the syllabuses of school certificate subjects and in adapting them to modern needs, and I think that we might devote some of the energy at present spent on arranging small modifications in the syllabuses to a general overhaul of the method of awarding school certificates. Although subjects such as housecraft, music, etc., have been introduced, they are treated mainly from a theoretical standpoint (if your wife is a good cook, pianist, etc., it is 10 to 1 she can't pass these examinations); the result is that girls, and, to a lesser extent, boys, who could give those children who are good at algebra, Latin, etc., a good hiding at dancing, music, art, housecraft, gardening, carpentry, etc., still have to spend twice as long as these other children on the homework of the theoretical subjects in addition to having to find time to fit in the extra classes for dancing, music, etc. It is a great credit to them that they can make any showing at all in the theoretical subjects, and they are often far worthier members of society than those to whom mathematics, classics, etc., come easily.

As far as I can see, the only way out of the impasse is to give every child a school certificate on leaving school, whatever his standard. This certificate would state:

(i) The subjects passed (with pass, credit or distinction) in an examination conducted like the present one, but under a new name, and modified so as to include a wider range of tests of craftsmanship and art. I call this the "secondary school test". When the leaving age is raised to sixteen every child should take this examination before leaving.

(ii) The child's prowess in any subject not included in the secondary school test—this would include any certificates in dancing, elocution and art awarded by national associations.

(iii) The child's prowess in sport, hobbies, etc.

(iv) The child's medical history, especially where illness or family troubles had interfered with schooling.

In order to make it possible to give this information accurately on the certificate, it would be an advantage if every child had a school report book in which the information required at the certificate stage was entered from year to year, and which could be passed from school to school, if the child moved.

The matriculation examination would then have to be absolutely separate from the secondary school test, and no certificate should be awarded on it; it should be purely an entrance examination to the universities, though exemption from it might still be granted to children who had passed an examination like the present Higher School Certificate.

This letter may seem rather out of place in the *Mathematical Gazette*. The bearing on mathematics is this: if you would welcome changes like those envisaged above, as I would, you must consider what the effect on mathematical teaching is going to be. I suggest that it will be twofold; first, most children up to the age of fifteen will spend less time on mathematics than at present, but they will learn what, for want of a better name, I call "practical mathematics"—they should, for instance, by fifteen be able to draw really

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practical graphs much better than most school certificate candidates now; secondly, the children, who are going in later on for subjects requiring mathematics, will, from thirteen or fourteen onwards, do more mathematics than the others, and, because they are segregated for part of the week, will make more rapid progress than at present.

Yours, etc., H. V. LOWRY.

APPROXIMATION.

To the Editor of the *Mathematical Gazette*.

SIR,—Mr. C. V. Durell wisely prefaces his chapter on approximations in his *General Arithmetic for Schools* with the maxim, "The important thing about approximate answers is that they should be right as far as they go."

On page iii of the Contents in the Board of Education's *Report on Curriculum and Examinations in Secondary Schools* we read: "Note.—The estimated gross cost of the preparation of the Appended Report (including the expenses of the witnesses and members of the Committee) is £1,735 17s. 7d., of which £280 represents the estimated cost of printing and publishing this Report."

If this figure is correct as far as it goes, it ceases to be an estimate; if an estimate it contains false figures. No wonder there is confusion in the mind of the Fourth Former who tries to reconcile school work and the outside world.

Yours, etc., N. E. BLAKE.

A GEOMETRICAL RECREATION.

To the Editor of the *Mathematical Gazette*.

SIR,—For a home-made near-mathematical recreation, consider the polyhedra whose vertices are the vertices of a given cuboid; count the differently-shaped (a) polyhedra, (b) tetrahedra, (c) pairs of mirror images.

In a race to give the answers real mathematicians would, of course, be bunched together as the winners, but fumblers may have to bestir themselves to avoid the disgrace of being beaten by expert potato slicers. For this is also a practical man's problem: his saws and knife-blades make passable planes, and he has abundant material to his hand in wood or stone, soap, cheese or root vegetables for making cuttable cuboids.

If the cutting be reckoned work, recreation must be sought in putting the pieces in place again. This, though undoubtedly an utterly infantile occupation, may give passing amusement to junior geometers, very childlike senior ones, some invalids, and even to a few mere doodlers.

It is not quite as easy as theoreticians may suppose. There is at least one carpenter who after cutting his block—under instruction—into six quadrangular tetrahedra had to resort to the base device of matching his saw-cuts before he could get them rightly together again.

This is the most interesting variety of cutting. Indeed, if several cuboids of almost, but not quite, identical dimensions be so cut; if the surfaces be given uniform appearance and the pieces mixed; and if—the largest *if* of all—a real mathematician can be induced to play with them, it may happen that—despite knowing every coordinator from Feodorov to Plücker—he will not instantly reconstitute the cuboids. The puzzle is called "The Six Plopps Puzzle".

Yours, etc., O. MADDEN.

MATHEMATICAL NOTES.

1709. *Notes on Conics. 10: Fontené's Theorem.*

§ 1. As Hardy has remarked, there is an element of caprice even in mathematical reputations. Feuerbach is immortalised by a theorem that he stumbled on in the course of the most pedestrian calculations, and for more than eighty years his theorem was an almost unintelligible curiosity; his immortality should be shared by Fontené, who dispelled the mystery in 1905 by enunciating a general theorem to which Feuerbach's is the most obvious of corollaries:

Th. 1. *The pedal circle of two isogonally conjugate points in a triangle touches the nine-point circle if and only if the points are collinear with the circumcentre.*

The extraordinary part of the story is that the mystery might have been dispelled seventy-six years earlier, for in 1829, that is, only seven years after Feuerbach's announcement of his result, Bobillier gave the theorem that

Th. 2. *The pedal circle of a point P for a triangle ABC passes through the centre of the rectangular hyperbola $ABCP$,*

to which the preceding Note in this series was devoted. Since it was already well known that

Th. 3. *The centre of any rectangular hyperbola round a triangle is on the nine-point circle,*
the deduction that

Th. 4. *If U, V are isogonally conjugate in ABC , their pedal circle cuts the nine-point circle in the centres of the rectangular hyperbolas $ABCU, ABCV$, could have been drawn at once, and Feuerbach's theorem follows no less readily from Th. 4 than from Th. 1. It was very soon after the publication of Th. 1 that Fontené himself pointed out the connection of his theorem with Bobillier's. If it is true that the bearing of Bobillier's theorem on Feuerbach's had not previously been noticed, this is surely the most astonishing failure to put two and two together in the whole history of mathematics.*

§ 2. Fontené first proves, and his sequence is reproduced by Coolidge and by Baker, that

Th. 5. *If a variable point U describes a line through the circumcentre, the pedal circle of U passes through a fixed point on the nine-point circle.*

The introduction of a varying point does nothing but confuse the argument. What we require, and what in fact Fontené gives us, is a static construction which associates with a single point U a point common to the pedal circle and the nine-point circle. To infer Feuerbach's theorem we need further to know that the construction when applied to the isogonal conjugate of U does not in general lead to the same point as when applied to U . This detail, the one logical necessity, is usually ignored; Coolidge, for example, says "The other intersection of the nine-point and pedal circles will be similarly obtained from the isogonal conjugate", and proceeds to Fontené's theorem without disposing of the logical alternative that would wreck the proof.

The risk of a fallacy is clear in our enunciation of Th. 4 above. Since U, V are not independent, it is not logically absurd to suggest that perhaps the two rectangular hyperbolas $ABCU, ABCV$ necessarily coincide. If this suggestion was confirmed, Th. 4 would locate only one of the two points of intersection of the circles in any case, and coincidence of the points U, V themselves would have no significance in relation to contact of the circles.

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Th. 6. *If U, V are isogonally conjugate in ABC , the rectangular hyperbolas $ABCU, ABCV$ are not necessarily identical that we can deduce from Th. 4 the result that*

Th. 7. *If U, V are isogonally conjugate in ABC , their pedal circle touches the nine-point circle if and only if the five points A, B, C, U, V are on a rectangular hyperbola.*

We shall see that Th. 7 is equivalent to Fontené's, but we can infer Feuerbach's theorem from Bobillier's without waiting to establish the equivalence.

§ 3. We take up now the systematic demonstration of our theorems, referring to the preceding Note only for Th. 2 itself. Th. 3, as was shown there, is a corollary of Th. 2, but in fact it is a much more primitive theorem, derivable from two elementary results :

Th. 8. *If $ABCP$ are four points on a circle whose centre is O , and if p is the pedal line of P for ABC , the four crosses $(PA, BC), (PB, CA), (PC, AB), (PO, p)$ have parallel axes ;*

Th. 9. *If q is a diameter of a rectangular hyperbola, and u is any line parallel to the chords bisected by q , the axes of the cross (q, u) are parallel to the asymptotes.*

If now A', B', C' are the midpoints of the sides of ABC , and if Q is the centre of a rectangular hyperbola through ABC , Th. 9 implies that the crosses $(QA', B'C'), (QB', C'A'), (QC', A'B')$ have parallel axes. Th. 3 follows immediately from Th. 8, or, to be strictly accurate, from the converse of the first part of Th. 8, and there is no gap to be filled between Th. 3 and Th. 4.

Only a particular instance is wanted to establish Th. 6 ; we may say for example that if U is in AO , then V is in AH , and the rectangular hyperbola $ABCV$ degenerates into the pair of lines AH, BC and does not contain U . This completes the proof of Th. 7, and incidentally of Feuerbach's theorem.

§ 4. But to understand the relation between Th. 7 and Fontené's simpler theorem, we must examine a little more closely the relation of isogonal conjugacy. A vertex of ABC is conjugate with every point on the opposite side, but otherwise the relation is one-one.

Let U be a variable point on a fixed line l , and let V be the isogonal conjugate of U . Then for any selection of points,

$$B(V) \wedge B(U) \wedge C(U) \wedge C(V).$$

Hence the points V lie on a fixed conic A through B, C , and this conic contains A also. Every point of A , unless it is a vertex of the triangle, leads back to a definite point of l .

Th. 10. *The isogonal transform of a line is a conic circumscribing the fundamental triangle.*

To the pencil of lines through a point U corresponds the pencil of conics through $ABCV$. In particular, lines through O correspond to conics through H :

Th. 11. *The rectangular hyperbolas circumscribing the fundamental triangle are the isogonal transforms of the diameters of the circumcircle.*

This result transforms Th. 7 into Th. 1, the theorem easier to prove into the theorem with the simpler enunciation.

§ 5. It remains to locate in elementary terms the points in which the pedal circle of an arbitrary point cuts the nine-point circle, and to confirm Fontené's

theorem by an argument which does not rely on an interpretation of coincidence. If Q , a point on the nine-point circle, is the centre of a rectangular hyperbola through ABC , this hyperbola cuts the circumcircle in the point R such that $HR = 2HQ$. The primitive theorem to be used is

Th. 12. *The isogonal conjugate of a point R on the circumcircle is the point at infinity on lines perpendicular to the pedal line of R .*

Since the isogonal transform of a line l contains the conjugate of the point at infinity on l , we have a construction, from the point U , for the centre of the rectangular hyperbola $ABCV$:

Th. 13. *If the lines through A, B, C perpendicular to a diameter l of the circumcircle cut the circle again in A_1, B_1, C_1 , the lines through A_1, B_1, C_1 perpendicular respectively to BC, CA, AB are concurrent in a point R of the circumcircle, and the pedal circle of any point of l cuts the nine-point circle midway between R and the orthocentre.*

This theorem implies Th. 5, as enunciated above in general terms; the construction is simpler than that by which Bricard gave the first geometrical proof of the result.

Th. 13 determines one of the points in which the pedal circle of a point U cuts the nine-point circle; the other of these points is the centre of the rectangular hyperbola $ABCU$.

Th. 14. *Of the two points in which the pedal circle of an arbitrary point U cuts the nine-point circle, one is given by the construction of Th. 13 applied to OU , and the other is a point common to the nine-point circles of the three triangles BCU, CAU, ABU .*

§ 6. If two points on a line l are isogonally conjugate, then each, as the conjugate of the other, lies also on the isogonal transform Λ . Omitting the case of self-conjugate points, which is highly special since the only self-conjugate points are the four contact centres, and not discriminating between real and imaginary intersections, we can say that

Th. 15. *On any line there is one pair of isogonally conjugate points, namely, the points of intersection of the line and its isogonal transform.*

Suppose now that l is a diameter of the circumcircle; let R be the significant point in which Λ cuts the circumcircle, let Q be the midpoint of HR , and let R' be the point of the circumcircle diametrically opposite to R . Then the pedal line of Q for $A'B'C'$ is parallel to the pedal line of R' for ABC , and is therefore perpendicular to the pedal line of R for ABC and parallel to l . It follows from Th. 8 that if N is the centre of the nine-point circle, the axes of the cross (QN, l) are parallel to the asymptotes of the hyperbola Λ . Hence from Th. 9 we have

Th. 16. *If l passes through the circumcentre, the diameter of Λ which bisects chords of Λ parallel to l passes through the nine-point centre, in direct verification of Fontené's theorem.*

[The references are to Coolidge's *Treatise on the Circle and the Sphere*, pp. 51 and 122, and to Baker's recent *Introduction to Plane Geometry*. Fontené's theorem was published in *Nouvelles Annales de Math.*, sér. 4, t. 5, p. 504; Coolidge reproduces a proof given by Bricard in the same journal, t. 6, p. 59. Bobillier's study of the rectangular hyperbola appeared in *Gergonne*, t. 19, p. 349, and Fontené's application of Bobillier's theorem in *Nouvelles Annales*, sér. 4, t. 6, p. 508.]

E. H. N.

1710.

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1710. Gramophone tracking error.

Tracking error is measured by the angle between the tangent to the groove and the vertical plane containing the needle. This angle should be zero if wear on the record and distortion in the sound produced are to be reduced to a minimum. In this note it is shown how the older type straight arms introduced considerable error, and how modern pick-ups with inclined heads or curved arms can be made to give almost perfect tracking.

1. Straight arm.

In Fig. 1, O is the centre of the record disc, P is the pivot of the arm, N the needle point. Let r_o be the radius of the outermost groove, r_i the radius of

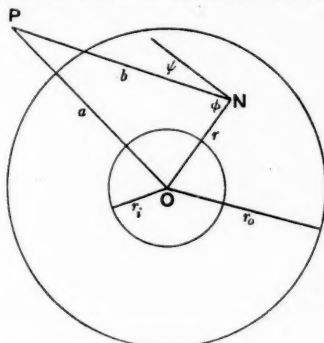


FIG. 1.

the innermost groove; and let $OP=a$, $PN=b$ (the length of the arm), $ON=r$ ($r_i \leq r \leq r_o$). Then for the tracking error ψ we have

$$\begin{aligned}\psi &= \frac{1}{2}\pi - \phi \\ &= \text{arc sin}\{(r^2 + b^2 - a^2)/2rb\}.\end{aligned}$$

On most gramophones with straight arms it will be found that $b=a$. In this case, $\psi = \text{arc sin}(r/2a)$.

For $a=r_o$ (the least possible value of a) the tracking error at the outside edge is therefore

$$\psi_o = \text{arc sin } \frac{1}{2} = 30^\circ. \dots\dots\dots(i)$$

For $a=r_o\sqrt{2}$, which will be taken throughout as a practical value of a in numerical calculations,

$$\psi_o = \text{arc sin}(1/2\sqrt{2}) = 20.7^\circ. \dots\dots\dots(ii)$$

By shortening the arm, that is, by reducing b , we may make $\psi_o = -\psi_i$,

that is, $\text{arc sin}\{(r_o^2 + b^2 - a^2)/2r_o b\} = -\text{arc sin}\{(r_i^2 + b^2 - a^2)/2r_i b\}$

or $(r_o^2 + b^2 - a^2)/2r_o b = -(r_i^2 + b^2 - a^2)/2r_i b$,

that is $r_o r_i (r_o + r_i) + (r_o + r_i)(b^2 - a^2) = 0$

or $r_o r_i + b^2 - a^2 = 0$,

which is an equation for a or b when the other three quantities are known.

An examination of record discs shows that in general r_i is approximately $\frac{1}{2}r_o$ and taking (as before) $a=r_o\sqrt{2}$, we have

$$\frac{1}{2}r_o^2 + b^2 - 2r_o^2 = 0,$$

from which we have $b=r_o\sqrt{\frac{5}{3}}$ for this case.

Using this value for b we have

$$\begin{aligned}\psi_0 &= \arcsin \{(r_o^2 + \frac{5}{3}r_o^2 - 2r_o^2)/2r_o^2\sqrt{\frac{5}{3}}\} \\ &= \arcsin(1/\sqrt{15}) \\ &= 14^\circ 58' \text{ or } 15^\circ \text{ very nearly.} \dots\dots\dots(iii)\end{aligned}$$

On evaluating $-\psi_i$ we get the same result. In this manner the maximum value of the tracking error is somewhat reduced.

Obviously there is one value of $r=r_c$ where the tracking is perfect, given by

$$r_c^2 = a^2 - b^2 = (2 - \frac{5}{3})r_o^2 = \frac{1}{3}r_o^2,$$

that is, $r_c = 0.58r_o$ approximately.

2. Curved arm or arm with inclined head.

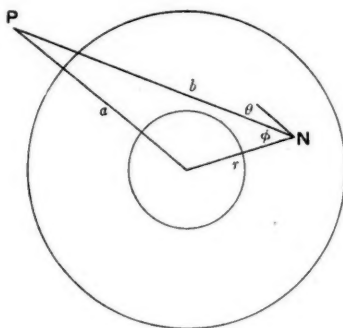


FIG. 2.

Using the notation of the previous figure, now let θ be the angle of inclination of the pick-up head to the line PN . Then

$$\psi = \frac{1}{2}\pi - (\theta + \phi)$$

and

$$\cos \phi = (r^2 + b^2 - a^2)/2rb, \text{ as before ;}$$

thus

$$\psi = \arcsin \{(r^2 + b^2 - a^2)/2rb\} - \theta.$$

In this case it is possible to make the tracking errors equal for $r=r_o$ and $r=r_i$, and of the same magnitude but to the other side of the groove for some value r_m of r between r_o and r_i ; that is,

$$\psi_0 = -\psi_m = \psi_i.$$

Clearly r_m gives a stationary value for ψ and so $d\psi/dr = 0$.

This implies that

$$2rb \cdot 2r - (r^2 + b^2 - a^2) \cdot 2b = 0,$$

whence

$$r_m^2 = b^2 - a^2.$$

We now have

$$\psi_0 = \arcsin \{(r_o^2 + b^2 - a^2)/2r_o b\} - \theta \dots\dots\dots(A)$$

$$= -\psi_m = \theta - \arcsin \frac{(b^2 - a^2) + b^2 - a^2}{2b\sqrt{(b^2 - a^2)}} = \theta - \arcsin \frac{\sqrt{(b^2 - a^2)}}{b} \dots\dots(B)$$

$$= \psi_i = \arcsin \{(r_i^2 + b^2 - a^2)/2r_i b\} - \theta. \dots\dots\dots(C)$$

From (A) and (C) we have

$$r_i(r_o^2 + b^2 - a^2) = r_o(r_i^2 + b^2 - a^2),$$

or, dividing through by $r_o - r_i$,

$$r_o r_i - b^2 + a^2 = 0, \dots\dots\dots (D)$$

which is an equation for a or b in terms of the other three quantities. From (A) and (B),

$$\arcsin \frac{r_o^2 + b^2 - a^2}{2r_o b} + \arcsin \frac{\sqrt{(b^2 - a^2)}}{b} = 2\theta.$$

Substituting for b and using (D),

$$\begin{aligned} 2\theta &= \arcsin \frac{r_o^2 + r_o r_i}{2r_o \sqrt{(a^2 + r_o r_i)}} + \arcsin \frac{\sqrt{(r_o r_i)}}{\sqrt{(a^2 + r_o r_i)}} \\ &= \arcsin \frac{r_o + r_i}{2 \sqrt{(a^2 + r_o r_i)}} + \arcsin \frac{\sqrt{(r_o r_i)}}{\sqrt{(a^2 + r_o r_i)}}. \dots\dots\dots (E) \end{aligned}$$

When

$$a = r_o \sqrt{2} \quad \text{and} \quad r_i = \frac{1}{3} r_o,$$

$$b^2 = a^2 + r_o r_i = \frac{7}{3} r_o^2$$

and

$$b = 1.5275 r_o.$$

Substituting these values of a and r_i in (E) we have

$$2\theta = \arcsin(2/\sqrt{21}) + \arcsin(1/\sqrt{7})$$

$$= 25^\circ 52.6' + 22^\circ 12.5' = 48^\circ 5.1',$$

whence

$$\theta = 24^\circ 2.5'.$$

From (A), (B) and (D),

$$2\psi_{\max} = \arcsin \frac{r_o + r_i}{2 \sqrt{(a^2 + r_o r_i)}} - \arcsin \frac{\sqrt{(r_o r_i)}}{\sqrt{(a^2 + r_o r_i)}} \dots\dots\dots (F)$$

$$= 25^\circ 52.6' - 22^\circ 12.5' = 3^\circ 40.1'$$

and

$$\psi_{\max} = 1^\circ 50'. \dots\dots\dots (iv)$$

This result may be compared with (ii) and (iii) in section 1, where the pivot P was in the same position.

Thus it is not a difficult matter to design a pick-up to give small tracking error. A value of a may be chosen and assumptions made as to the values of r_o and r_i ; for example, for a twelve-inch record, $r_o = 5.75$ inches, $r_i = 1.75$ inches. Then b and θ can be calculated from equations (D) and (E), and the resulting maximum tracking error from (F).

It seems reasonable to assume that the total wear on the record due to tracking error is proportional to

$$W = \int_{r_i}^{r_o} |\psi| dr,$$

at least for small ψ , and it may be desirable to design the pick-up so as to make W a minimum rather than to proceed as above. In order to find $dW/d\theta$ it is necessary to know r_e' and r_e'' , the values of r for which $\psi = 0$, that is, for which

$$\arcsin \{(r^2 + b^2 - a^2)/2rb\} - \theta = 0,$$

or

$$r^2 - 2br \sin \theta + b^2 - a^2 = 0,$$

whence

$$r_e = b \sin \theta \pm \sqrt{(a^2 - b^2 \cos^2 \theta)}.$$

For a change $\delta\theta$ in the inclination of the head (noting that W is the integral of the modulus of ψ , and that $\delta\theta = \delta\psi$), we have

$$\delta W = \{(r'_c - r_i) - (r''_c - r'_c) + (r_o - r''_c)\} \delta\theta$$

or

$$\begin{aligned} dW/d\theta &= r_o - r_i + 2(r'_c - r''_c) \\ &= r_o - r_i - 4\sqrt{(a^2 - b^2 \cos^2 \theta)} \\ &= r_o - r_i - 4\sqrt{(a^2 - (a^2 + r_o r_i) \cos^2 \theta)}, \end{aligned}$$

and if $dW/d\theta = 0$ this gives

$$\cos^2 \theta = \{16a^2 - (r_o - r_i)^2\} / 16(a^2 + r_o r_i).$$

For

$$a = r_o \sqrt{2}, \quad r_i = \frac{1}{3} r_o \text{ as before, } \theta = 23^\circ 0' 32'.$$

Tracking error at outside and inside : $\psi_o = \psi_i = 2^\circ 52' 3'' < 2.9^\circ$.

Maximum tracking error near middle : $\psi_m = 0^\circ 47' 8'' < 0.8^\circ$.

Thus we see that the average tracking error has been considerably reduced at the expense of allowing an increase at the ends.

Auckland University College, N.Z.

N. J. RUMSEY.

1711. *An example in abstract mathematics.*

1. It is not easy to give a non-trivial example of a piece of reasoning in mathematics which can be presented, from the outset, in a purely symbolic way. If, say, algebra or geometry is taken, and the reader is requested to view the work abstractly, some shreds of the old notions may still survive in his mind. I here give a short theory in which the "meaning" is carefully concealed from the start. Only one basic law is assumed.

2. Consider a set of elements a, b, c, \dots of any kind such that any two, in a particular order, produce an element in the set; if a, b are elements which so combine, the element which they produce, and which is supposed always to exist and be unique, is denoted by ab . Thus ab is an element of the class, and so is ba , which may, or may not, be the same as ab . Of course, nothing which has been said prevents ab from being a or b ; and as a special case, a may combine with itself, producing aa . Thus from a, b, c, d we get, for example, the elements $((ab)(cd))$, $(a(bc))d$, $a((bc)d)$, and so on, and these may, or may not, be the same. We use the sign of equality to indicate identity of elements, and for convenience we write $abcd \dots$ instead of $((((ab)c)d) \dots)$.

Hence if x, y be any combination of elements, and if it has been proved that $x = y$, it will follow that $xa = ya$, where a is any element, but it will not follow that $ax = ay$, since the first step in forming ax is the combining of a with the first factor in x , and this need not be the same as the first factor in y .

3. *The only law we shall assume is*

$$abcab = c, \dots\dots\dots(i)$$

where a, b, c are any elements, denoted by expressions of any degree of complication.

We shall deduce

$$aa = bb.$$

In (i) substitute abd for a ; a for b ; b for c . Then

$$(abd)ab(abd)a = b,$$

that is

$$abdab(abd)a = b.$$

Hence, by (i),

$$d(abd)a = b,$$

and hence

$$d(abd)ac = bc.$$

Substituting ac for d ,

$$(ac)(ab(ac))ac = bc,$$

that is

$$ac(ab(ac))ac = bc.$$

Hence, by (i),

$$ab(ac) = bc. \dots\dots\dots(ii)$$

Hence

$$ab(ac)ab = bcab.$$

Hence, by (i)

$$ac = bcab. \dots\dots\dots(iii)$$

Hence

$$acc = bcabc,$$

and by (i)

$$= a.$$

Thence

$$acc = a, \quad caa = c. \dots\dots\dots(iv)$$

Thus we can drop a repeated letter, unbracketed, from the end of any expression. Hence (iii) and (iv) give

$$acb = bcabb,$$

$$acb = bca. \dots\dots\dots(v)$$

This with (iv) gives

$$cca = a, \quad ccaa = aa.$$

Hence by (iv)

$$cc = aa, \text{ as desired.}$$

4. We can proceed a little further, writing u for aa , bb , We must note that au stands for $a(bb)$, for example, and hence is not necessarily equal to abb , which is equal to a . But we do have $ua = a$. Now in (ii) put $c = a$; then

$$abu = ba. \dots\dots\dots(vi)$$

Hence

$$b(aub) = (aub)bu = (aubb)u = (au)u = auu = a. \dots\dots\dots(vii)$$

By (v),

$$aub = bua. \dots\dots\dots(viii)$$

By (vi), (v)

$$abuc = bac = cab = acub.$$

Writing au for a ,

$$aubuc = aucub.$$

Hence, by (viii)

$$buauc = bu(auc),$$

which we rewrite as

$$aubuc = au(buc). \dots\dots\dots(ix)$$

The reader will now be able to guess an "interpretation" from (vii), (viii), (ix).

5. If we assume, not only (i), but also that our "products" obey the associative law, we get $a \cdot aa = aa \cdot a$, and hence $au = ua = a$. Then (vi) gives

$$ba = abu = ab;$$

thus the commutative law follows.

Further,

$$aub = abu = ab;$$

the interpretation is again left to the reader.

6. As an example of "metatheory" (theory *about* the theory), it is easy to show that any formula which can be deduced from (i) is such that the number of occurrences of any letter is of the same parity for each side of the formula; (u of course represents two letters). The simplest law which violates this is $aa = a$. If we adjoin that, our class contains only one element.

H. G. FORDER.

1712. A Scholarship question.

The question (i) starred by Mr. A. Robson in the *Gazette* (October, 1942) is the case for order three of a general theorem (or lemma) of Minkowski's.

If a_{ii} be positive, and a_{ik} negative when $i \neq k$ for $i, k = 1, 2, \dots n$, and if

$$\sum_{k=1}^n a_{ik} > 0,$$

then the determinant of the a_{ik} cannot vanish.

Minkowski gives a proof by induction, involving an expansion of the determinant. Artin makes a remark in *Crelle*, 167, which might easily be overlooked (and this is the reason for this note), that it is generally believed that the proof of Minkowski's theorem is difficult but that actually it is quite simple:

If the determinant vanishes, then there are quantities x_k ($k=1, \dots, n$), not all zero, such that

$$\sum_{k=1}^n a_{ik} x_k = 0, \quad (i=1, \dots, n).$$

Let $|x_i|$ be the maximum among $|x_1|, \dots, |x_n|$, then the i th equation

$$a_{ii} x_i + \sum_{k \neq i} a_{ik} x_k = 0$$

contradicts the hypothesis, since $a_{ii} x_i$ cannot cancel the rest.

H. G. FORDER.

1713. Note on Turner's Theorem.

Professor Turner's theorem which (see *Math. Gazette*, Vol. XX, p. 182) he conjectured on the basis of careful drawings in 1913 and which sixteen years later when a tiresomely long proof had been found he communicated to the American Association, runs as follows: If three ellipses are such that each pair has one common focus and two real points of intersection, the three chords of visible intersection are concurrent. Professor Neville and Mr. B. E. Lawrence have published proofs of the theorem (*vide supra* and Notes 1237 and 1238). The proof below is I think simpler than any others I have seen. The algebra is real throughout.

If r_1, r_2, r_3 are the distances of a point from O_1, O_2, O_3 , then for points on the ellipse S_2 , foci O_2, O_1 , major axis b ,

$$r_3 + r_1 = b > O_3 O_1$$

and so

$$2br_1 = r_1^2 + b^2 - r_3^2.$$

For points on the ellipse S_3 , foci O_1, O_2 , major axis c ,

$$2cr_1 = r_1^2 + c^2 - r_2^2.$$

Hence the common points of S_2 and S_3 , if they intersect, must satisfy

$$b(r_1^2 + c^2 - r_2^2) = c(r_1^2 + b^2 - r_3^2)$$

or

$$l_1 \equiv (b-c)r_1^2 - br_2^2 + cr_3^2 - bc(b-c) = 0.$$

But by the extension of Apollonius' Theorem all points satisfying this relation lie on a straight line, which must be the common chord of S_2 and S_3 , if they intersect (and incidentally they cannot therefore intersect in more than two points).

Similarly the other two common chords are

$$l_2 \equiv ar_1^2 + (c-a)r_2^2 - cr_3^2 - ca(c-a) = 0$$

and

$$l_3 \equiv -ar_1^2 + br_2^2 + (a-b)r_3^2 - ab(a-b) = 0$$

since

$$a(b+c-a)l_1 + b(c+a-b)l_2 + c(a+b-c)l_3 \equiv 0,$$

the point which satisfies $l_2=0$ and $l_3=0$ will satisfy $l_1=0$, in other words the three common chords are concurrent.

Hyperbolae may be substituted for one or more of the three ellipses provided that the three common chords are chosen so that any pair of them that are chords of the same hyperbola are chords on different branches of that hyperbola. The reason for this is that

$$2br_1 = r_1^2 + b^2 - r_3^2 \quad \text{and} \quad 2br_3 = r_3^2 + b^2 - r_1^2, \quad |b| < O_1 O_3,$$

represent different branches of the hyperbola.

The same analysis also gives a theorem in 3-dimensional geometry about focus-sharing ellipsoids of revolution intersecting in plane curves whose planes have a common line of intersection.
R. C. LYNESS.

1714. *Integral cyclic hexagons.*

The cyclic hexagon with sides and diagonals of integral length given in Note 1514 of the *Gazette* (XXV, p. 113) is only one of an infinite family of such hexagons. Let p be any integer such that integers a and b can be found to satisfy the equation $p^2 = a^2 + b^2$. Then

$$p^4 = (a^2 + b^2)^2 = (pa)^2 + (pb)^2 \\ = (a^2 - b^2)^2 + (2ab)^2.$$

Thus we have two pairs of integers x and y which satisfy the equation $p^4 = x^2 + y^2$, it being readily proved that the pairs of values $(pa, pb)(2ab, a^2 - b^2)$ are different.

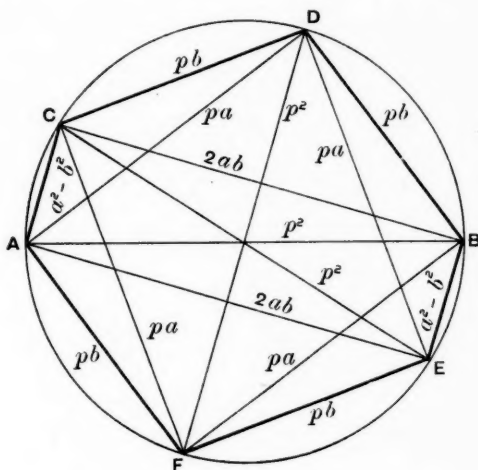


FIG. 1.

Let the two right-angled triangles with sides

$$p^2, pa, pb; \quad p^2, 2ab, a^2 - b^2$$

be fitted into a semi-circle as shown in Fig. 1. Then, using Ptolemy's theorem,

$$AB \cdot CD + AC \cdot BD = AD \cdot BC,$$

that is,

$$p^2 \cdot CD + (a^2 - b^2)pb = 2ab \cdot pa$$

and hence

$$CD = pb,$$

and we have a cyclic hexagon, as illustrated in Fig. 1, in which all sides and diagonals are of integral length.

In particular, integers a and b can be found to satisfy $p^2 = a^2 + b^2$, where p is a prime number of the form $4n + 1$, where n is an integer, or if p is of this form, p^2 may be expressed as the sum of the squares of two integers.

Dr. Langford has taken $p = 5$, the lowest of such primes, and so in his hexagon the sides have the smallest integral values obtainable in hexagons of this kind.

Another type of cyclic hexagon with sides and diagonals of integral length may be constructed using the following algebraic analysis. Let us consider the pairs of integral values of x and y which satisfy the equation $p^2q^2 = x^2 + y^2$, where $p^2 = a^2 + b^2$, $q^2 = c^2 + d^2$, and p, q, a, b, c, d are integers.

$$\begin{aligned} p^2q^2 &= p^2(c^2 + d^2) = (pc)^2 + (pd)^2 \\ &= q^2(a^2 + b^2) = (qa)^2 + (qb)^2. \end{aligned}$$

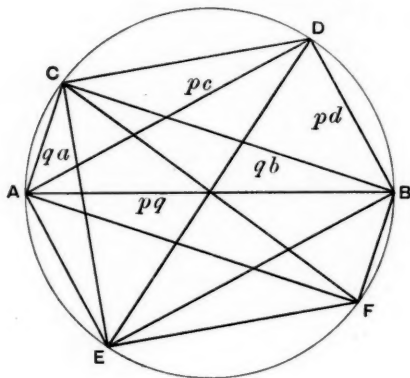


FIG. 2.

Let rectangles with sides pc, pd ; qa, qb be inscribed in a circle as shown in Fig. 2.

Then

$$AB \cdot CD = AD \cdot BC - AC \cdot BD,$$

whence

$$pq \cdot CD = pq(bc - ad),$$

and $CD = bc - ad$ and is of integral length.

$$DE^2 = p^2q^2 = CD^2 + CE^2$$

whence

$$\begin{aligned} CE^2 &= (pq)^2 - (bc - ad)^2 \\ &= (ac + bd)^2, \end{aligned}$$

and CE is also integral in length. Hence all the sides and diagonals of the hexagon $ACDBFE$ are of integral length.

If AD were drawn of length pd and hence BD of length pc , then, proceeding as above, CD is shown to be of length $bd - ac$, an integral value, and CE of length $bc + ad$, an integral value. Hence we find a second hexagon which can be inscribed in the same circle, having all sides and diagonals of integral length.

As an illustration, take the case $p = 5, q = 13$;

$$\begin{aligned} p^2q^2 &= 65^2 = (13 \cdot 4)^2 + (13 \cdot 3)^2 \\ &= (5 \cdot 5)^2 + (5 \cdot 12)^2 \\ &= 56^2 + 33^2 \\ &= 63^2 + 16^2. \end{aligned}$$

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(47, 1104), (576, 943), (264, 1073), and (744, 817), are such that x and y are relatively prime to each other and to 1105, and the learned author by his remark does not seem to do Liouville justice.

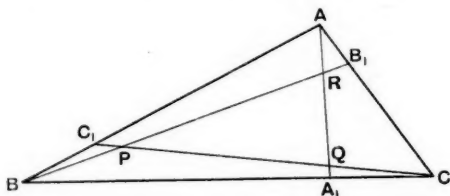
(2) If $z = p \cdot q$, p and q different primes of form $4n+1$, the equation $x^2 + y^2 = z^2$ (x, y and z integers) has 4 solutions, 2 of which give relatively prime values for x, y and z ; if $z = p \cdot q \cdot r$, p, q, r primes of form $4n+1$ and all different, the equation has 13 solutions, 4 of which give relatively prime values for x, y and z ; and generally, if $z = p \cdot q \cdot r \dots$, p, q, r, \dots m different primes of form $4n+1$, the equation has ${}^mC_1 + 2 \cdot {}^mC_2 + \dots + 2^{m-1} \cdot {}^mC_m$ solutions, 2^{m-1} of which give relatively prime values for x, y , and z .

(3) If $z = p^2$, p a prime of form $4n+1$, the equation $z^2 = x^2 + y^2$ (x, y and z integers) has 2 solutions, one of which gives relatively prime values x, y and z ; if $z = p^3$, p a prime of form $4n+1$, the equation has 3 solutions, one of which gives relatively prime values x, y and z ; and generally, if $z = p^m$, p a prime of form $4n+1$, the equation has m solutions, one of which gives relatively prime values for x, y and z .

P. C. WICKENS.

1715. Triangle properties.

If A_1, B_1, C_1 divide the sides of a triangle ABC in the ratio $m:n$ and AA_1, BB_1, CC_1 form the triangle PQR , then the triangles ABC, PQR have equal Brocard angles and the same centroid.



Other properties can be obtained; for example, if $m:n=2:1$, and $A=60^\circ$, $B=90^\circ$, $C=30^\circ$, then $P=60^\circ$, $Q=30^\circ$, $R=90^\circ$. This makes an amusing drawing exercise for children.

These properties may be added to those given by Mr. C. H. Hardingham in Note 1286, *Gazette*, XXII, May 1938, p. 184; I am indebted to the Editor for supplying me with this reference, since I was unaware of Mr. Hardingham's work when preparing the present note.

J. STORR-BEST.

1716. On Note 1636.

Is not a modified form of Hutton's formula much quicker and easier to handle: thus:

If x_n is an approximation to a root of

$$x^3 + bx + c = 0,$$

then a better approximation is

$$x_n - \{x_n f(x_n) / (2x_n^3 - c)\}$$

where the error is

$$(bh^2 + 2x_n h^3 + h^4) / (2x_n^3 - c),$$

in defect when the approximation x_n is less than the correct value of the root and *vice versa*. When x_n is a close approximation, the error term may be taken as

$$bh^2 / (2x_n^3 - c).$$

Taking the equation $x^3 - 2x - 5 = 0$, then with $x_1 = 2$, we have $x_2 = 2.095$, and then $x_3 = 2.0945515$, and correcting this by the error term we have 2.094551483 , correct to 9 figures.

Alternatively, if a is an approximation to a root of

$$x^3 + bx + c = 0,$$

and if a' is a first approximation obtained by the modified form of Hutton's formula, taking into account the error term, then a better approximation is

$$a - [a'f(a)/\{aa'(a + a') - c\}],$$

where a' can be chosen either above or below the true root and a can be the simplest choice. Thus if in the equation

$$x^3 - 2x - 5 = 0$$

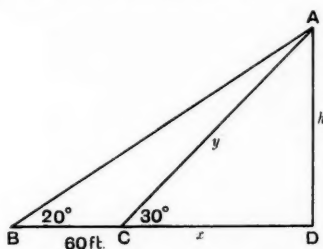
we take $a = 2.1$, $a' = 2.094$, then we get the root correct to 7 figures.

R. H. BIRCH.

1717. *A point in elementary trigonometry.*

I have looked in a number of textbooks and also asked a number of practical men about problems of this type:

A man observes that the angle of elevation of the top of a tower is 20° ; he walks 60 ft. directly towards the tower and sees that this angle is now 30° . Find the height of the tower. (The figure is not to scale.)



Many textbooks actually give an example like this as a worked example using the tangents of 20° and 30° , followed by the elimination of x by a method that many pupils find difficult.

Yet the practical users of trigonometry all did it as follows.

$$\angle BAC = 30^\circ - 20^\circ = 10^\circ;$$

$$h/y = \sin 30^\circ, \quad 60 \text{ ft.}/\sin 10^\circ = y/\sin 20^\circ.$$

Hence

$$h = \frac{\sin 30^\circ \sin 20^\circ}{\sin 10^\circ} \cdot 60 \text{ ft.}$$

Now, how can this anomaly be explained. Presumably the authors either want to show pupils as soon as possible that trigonometry can do all sorts of things they had not thought possible, or they are trying to find other examples to make the pupil familiar with the use of the tangent. But does either reason justify asking a pupil to do a problem by a method that would not be used by the practical man, particularly when it is sufficiently awkward to put quite a number of pupils off in despair? Surely it would be better to give more simple examples for practice and pass on as rapidly as possible to the sine rule when problems of this kind can then come in.

I should be glad to have other views on this point.

C. D. L.

1718. *On Note 1539.*

A model of the five interpenetrating tetrahedra mentioned by Mr. Cadwell may be made by taking advantage of the fact mentioned by Prof. Coxeter

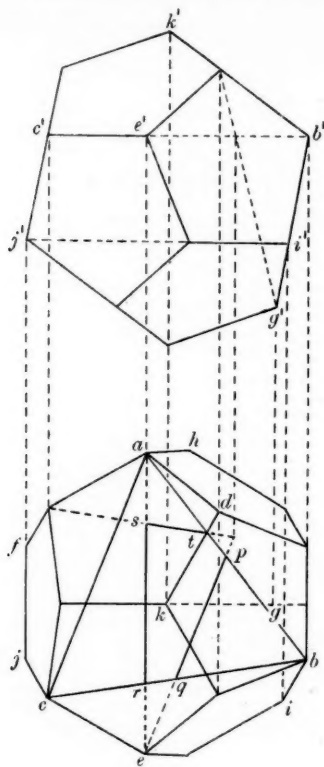


FIG. 1.

(*Ball*, p. 134) that the vertices of the tetrahedra are also vertices of a dodecahedron, and their faces are also the faces of an icosahedron.

Project the dodecahedron upon a face of an icosahedron, as in Fig. 1. Then abc in the horizontal projection will be a face of a tetrahedron. Take any four vertices, say $DEFG$, forming a tetrahedron, and by the usual methods find the horizontal traces of three of its faces: thus pq is the trace of the face DEG , rs the trace of DEF , and st of DFG .

An axis of trigonal symmetry emerges through the centre of each face of a tetrahedron: therefore reproduce the drawing of the triangle abc and the lines pq , rs , st on a separate sheet of paper and also on tracing paper. Rotate the tracing through 120° from its initial position and draw the lines pq , rs , st in their new situations: rotate the tracing through a further 120°

and repeat the operation (Fig. 2). Complete Fig. 2 by drawing the triangle HIJ representing the horizontal traces of the tetrahedron which has the face hij (Fig. 1) parallel to the plane of projection (and to abc). Fig. 2 gives the direction and length of all the edges of the model.

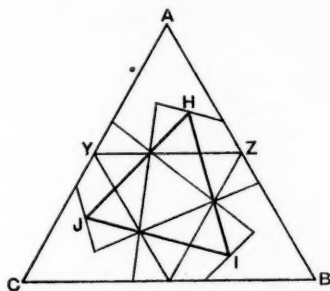


FIG. 2.

The first step in the practical construction consists in making a net of a regular tetrahedron of which ABC (Fig. 2) is one face. Reproduce all the lines within ABC on each face, and make up the tetrahedron. Having done this it may seem natural to go round the tetrahedron adding pieces to complete a second tetrahedron, and then a third, and so on : but in this way small errors accumulate and make themselves felt by the time the fourth tetrahedron is begun. It is advisable to adopt the following procedure whereby the general symmetry of the whole model is better preserved : Prepare four nets of a regular tetrahedron of which AYZ is one face (only three faces are required), and draw on each face all the lines within AYZ . Make up the tetrahedra and fix one in the position HIJ on each face of the initial tetrahedron. The model is then completed by adding trihedral solid angles as indicated by the lines on the faces. The lines in question are all comprised within the triangle AYZ and it can be seen by inspection of the model which of those lines is to be taken to complete the faces of any particular solid angle.

SIDNEY MELMORE.

1719. Pythagorean angles.

Arising out of the "suspicion" expressed at the end of Note 1624 (*Gazette*, XXVI, p. 186) that "Any Pythagorean angle can be expressed as the sum or difference of two other Pythagorean angles", the following investigation was undertaken.

These facts resulted :

- I. The difference of two Pythagorean angles is a Pythagorean angle.
- II. The sum of two Pythagorean angles is either a Pythagorean angle or the supplement of a Pythagorean angle.
- III. Any Pythagorean angle can be expressed, in an infinite number of ways, as the sum or difference of two other Pythagorean angles.

Let (a, b, c) denote a Pythagorean set ; that is, a, b and c are integers such that $a^2 + b^2 = c^2$. We shall suppose that a, b, c are prime to one another, and that $a < b$. Let α denote any Pythagorean angle ; $\alpha = \arctan(a/b)$ or $\arctan(b/a)$.

Let
so that

$$\theta = \text{arc tan } (a/b), \quad \phi = \text{arc tan } (b/a),$$

$$\theta < \frac{1}{4}\pi < \phi.$$

Thus the class α includes the classes θ and ϕ . Consider two sets, say p and q , where $\theta_p > \theta_q$. Use will be made of the identity

$$(a_p b_q + a_q b_p)^2 + (b_p b_q - a_p a_q)^2 = (a_p b_q - a_q b_p)^2 + (b_p b_q + a_p a_q)^2$$

$$= (c_p c_q)^2. \dots\dots\dots(1)$$

Now

$$\theta_p + \theta_q = \text{arc tan } (a_p/b_p) + \text{arc tan } (a_q/b_q)$$

$$= \text{arc tan } \{(a_p b_q + a_q b_p)/(b_p b_q - a_p a_q)\}.$$

By (1) this latter angle is an α , say α_s .

If

$$\theta_p + \theta_q < \frac{1}{4}\pi,$$

then,

$$a_p b_q + a_q b_p = a_s, \quad b_p b_q - a_p a_q = b_s, \quad \text{and} \quad \alpha_s = \theta_s.$$

If

$$\theta_p + \theta_q > \frac{1}{4}\pi,$$

then a_s and b_s are interchanged, and $\alpha_s = \phi_s$.

In either case, however, the result can be stated in the form

$$\theta_p + \theta_q = \alpha_s. \dots\dots\dots(2)$$

Similarly

$$\theta_p - \theta_q = \text{arc tan } \{(a_p b_q - a_q b_p)/(b_p b_q + a_p a_q)\}.$$

From (1), this angle is also an α , but, being smaller than θ_p it is also a θ , say θ_t .

So

$$a_p b_q - a_q b_p = a_t \quad \text{and} \quad b_p b_q + a_p a_q = b_t.$$

Also

$$\theta_p - \theta_q = \theta_t. \dots\dots\dots(3)$$

Now

$$\phi_p + \phi_q = \frac{1}{2}\pi - \theta_p + \frac{1}{2}\pi - \theta_q = \pi - \alpha_s, \dots\dots\dots(4)$$

$$\theta_p + \phi_q = \theta_p + \frac{1}{2}\pi - \theta_q = \frac{1}{2}\pi + \theta_t = \pi - \phi_t. \dots\dots\dots(5)$$

Also

$$\phi_p + \theta_q = \phi_t, \dots\dots\dots(6)$$

$$\phi_q - \phi_p = \theta_t, \dots\dots\dots(7)$$

$$\phi_p - \theta_q = \frac{1}{2}\pi - \theta_p - \theta_q = \frac{1}{2}\pi - \alpha_s,$$

and as $\frac{1}{2}\pi - \alpha_s$ is also an α , we can write

$$\phi_p - \theta_q = \alpha_s'. \dots\dots\dots(8)$$

Equations (3), (7) and (8) lead to I, and (2), (4), (5) and (6) to II.

From (3), (7) and (8) we have

$$\theta_p = \theta_q + \theta_t \dots\dots\dots(9)$$

$$\phi_q = \phi_p + \theta_t, \dots\dots\dots(10)$$

$$\phi_p = \theta_q + \alpha_s'. \dots\dots\dots(11)$$

These equations mean that any α can be expressed as the sum of two other α 's. Equation (9) gives the obvious incidental result that, if the α is a θ , then the other two α 's must be θ 's. Equations (10) and (11) together show that a ϕ may be expressed either as a sum of two θ 's or as the sum of a θ and a ϕ .

For a given θ_p we can find any number of corresponding angles θ_q (see proof below) so that θ_p can be expressed as the sum of two other θ 's in an infinite number of ways; similarly for ϕ_q and ϕ_p .

Again, from (2) and (6) we have

$$\theta_p = \alpha_s - \theta_q, \dots\dots\dots(12)$$

$$\phi_p = \phi_t - \theta_q. \dots\dots\dots(13)$$

Here we have the result that an α can be expressed as the difference of two other α 's. For a given θ_p , we can choose any $\alpha_s > \theta_p$ and find the corresponding θ_q . Similarly, for a given ϕ_p we can choose any $\phi_t > \phi_p$. Since the α_s (or ϕ_t) can be chosen in any number of ways, an α can be expressed in an infinite number of ways as the difference of two other α 's. Thus III above is established.

A special case occurs when $p = q$. Then

$$\begin{aligned}\theta_p + \theta_q = 2\theta_p &= \arctan \{(2a_p b_p) / (b_p^2 - a_p^2)\}, \\ \phi_p - \theta_q = \frac{1}{2}\pi - 2\theta_p &= \arctan \{(b_p^2 - a_p^2) / 2a_p b_p\}.\end{aligned}$$

It follows that, when n is an integer, $n\theta_p$ is an α provided that $n\theta_p < \frac{1}{2}\pi$. Thus α/n must in certain cases be a θ . When $n = 2$, we have

$$\frac{1}{2} \arctan (a/b) = \arctan \{(c-b)/a\}.$$

A necessary condition for this angle to be a θ is that b should be odd. A necessary and sufficient condition is that $2c(c-b)$ should be a perfect square. This is so for the set (336, 527, 625). If, however, we start with a ϕ , a and b are interchanged. (7, 24, 25) is such a case.

The phrase "in an infinite number of ways" in III above is based on the act that "given any θ_p , it is always possible to find a θ_q , ($\theta_q < \theta_p$), and given a ϕ_p , it is always possible to find a ϕ_q , ($\phi_q > \phi_p$). If one can be found, then an infinite number can be found. The truth stated is demonstrable as follows.

The Pythagorean set (a, b, c) can be represented by

$$a = mn, \quad b = \frac{1}{2}(m^2 - n^2), \quad c = \frac{1}{2}(m^2 + n^2),$$

where m, n are odd integers, prime to each other, $m > n$, when $m > n(1 + \sqrt{2})$. When $m < n(1 + \sqrt{2})$, a and b are interchanged.

We have to prove that it is possible to find $\theta_q < \theta_p$, that is,

$$\arctan \{2m_q n_q / (m_q^2 - n_q^2)\} < \arctan \{2m_p n_p / (m_p^2 - n_p^2)\},$$

$$\text{or} \quad 2 \arctan (n_q / m_q) < 2 \arctan (n_p / m_p),$$

$$\text{which reduces to} \quad n_q / m_q < n_p / m_p,$$

which is always possible.

When a and b are interchanged, the condition obtained is reversed, and this is also always possible. Therefore the result is proved for θ_p .

$$\text{Now if} \quad \theta_q < \theta_p, \quad \text{then} \quad \phi_q > \phi_p,$$

which establishes the result for ϕ_p .

SYDNEY THOMSON.

1720. Proof of rules for approximate division.

Here is a very simple method of proving any of the common rules for approximate division. It has the additional advantage of exhibiting at the same time the relative error in the approximate quotient.

To prove a rule for approximate division by N , we divide N by N using the approximate method. We know that the true value of the quotient is unity, and therefore any deviation from unity here is the relative error introduced by the approximate process.

As an illustration take the "Third, Tenth and Tenth" rule for division by 73. Apply the rule to divide 73 by 73, thus

3	.73
10	.24333
10	.02433
	.00243
	1.0001

which proves the rule and shows that the relative error is .0001 in excess.

R. HAMILTON DICK.

1721. *Areas of similar triangles.*

The (at present) fashionable proof that the areas of similar triangles are proportional to the squares on corresponding sides seems to be largely devoted to proving, in effect, that the sine of an angle is constant. Why not assume this and use trigonometry? Thus,

Given two similar triangles ABC, DEF

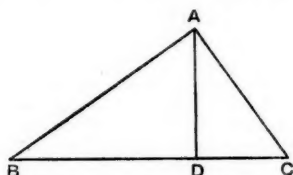
$$\frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}de \sin F} = \frac{a^2}{d^2} = \frac{b^2}{e^2} = \frac{c^2}{f^2},$$

since $\angle C = \angle F$ and $a/d = b/e = c/f$.

R. HAMILTON DICK.

1722. *On Notes 1550, 1597.*

A still simpler approach to the Theorem of Pythagoras is as follows :



From the similar triangles ABC, DAC, DBA ,

$$\begin{aligned} \frac{\Delta ABC}{a^2} &= \frac{\Delta DAC}{b^2} = \frac{\Delta DBA}{c^2} \\ &= \frac{\Delta DAC + \Delta DBA}{b^2 + c^2} \\ &= \frac{\Delta ABC}{b^2 + c^2}. \end{aligned}$$

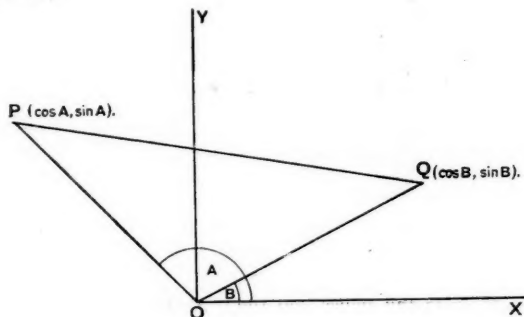
Hence

$$a^2 = b^2 + c^2.$$

R. HAMILTON DICK.

1723. *The addition formulae.*

Let OX, OY be the coordinate axes ; $\angle XOP = A$, $\angle XOQ = B$, so that $\angle POQ = (A - B)$.



Let OP , OQ be unit lengths, so that the coordinates of P are $(\cos A, \sin A)$ and those of Q are $(\cos B, \sin B)$.

$$\begin{aligned} PQ^2 &= (\cos B - \cos A)^2 + (\sin B - \sin A)^2 \\ &= 2 - 2 \cos A \cos B - 2 \sin A \sin B. \end{aligned}$$

Applying the cosine rule to the triangle OPQ , we get

$$\begin{aligned} \cos(A - B) &= (OQ^2 + OP^2 - PQ^2)/2 \cdot OQ \cdot OP \\ &= 1 - (1 - \cos A \cos B - \sin A \sin B) \\ &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

Then $\cos(A + B)$ is deduced from this formula.

$$\begin{aligned} \text{Further, } \sin(A - B) &= -\cos\{90^\circ + (A - B)\} \\ &= -\cos\{(90^\circ + A) - B\} \\ &= -\{\cos(90^\circ + A) \cos B + \sin(90^\circ + A) \sin B\} \end{aligned}$$

from the previous result.

Thus $\sin(A - B) = \sin A \cos B - \cos A \sin B$, and again the formula for $\sin(A + B)$ is deduced from this.

H. L. SHARMAN.

1724. "A minus and a minus make a plus." (See Note 1622.)

While it is usually easy to get pupils to accept and to use the rules that "a minus and a plus make a minus" as do a "plus and a minus", it is not so with the rule given above. The following aids may be of use.

(a) Taking the example given in Note 1622, we have

$$\begin{aligned} 10 - 5 &= 5, \\ \text{but } 10 - 5 &= 10 - (7 - 2) \\ &= 10 - 7 + 2 \text{ or } 10 - 7 - 2 \\ &= 12 - 7 \text{ or } 10 - 9 \\ &= 5 \text{ or } 1. \end{aligned}$$

Therefore only by using the rule given above do we obtain the right answer.

(b) $(-4)(-5)$. The numerical value of the product is 20 and its sign is either + or -.

$$\begin{aligned} \text{Now } (-4)(-5) &= (-4)(-7 + 2) \\ &= (-4)(-7) + (-4)(+2) \\ &= \text{either } +28 - 8 \\ &\quad \text{or } -28 - 8 \\ &= +20 \text{ or } -36. \end{aligned}$$

We see therefore that unless we agree that "a minus and a minus make a plus" we shall obtain absurd results; such as $(-4)(-5) = -36$.

(c) Suppose it had been agreed that, say, $(-2)(+3) = -6$.

$(-2)(-3) = -6$ or $+6$. Now if both $(-2)(+3)$ and $(-2)(-3) = -6$, we have $(-2)(+3) = (-2)(-3)$, therefore $+3$ would be equal to -3 .

The conclusion is the same as that in (b).

$$\begin{aligned} (d) \text{ "This is a house that Jack built."} & \quad (+)(+) = (+) \\ \text{ "This is a house that Jack did not build."} & \quad (+)(-) = (-) \\ \text{ "This is not a house that Jack built."} & \quad (-)(+) = (-) \\ \text{ "This is not a house that Jack did not build."} & \quad (-)(-) = (+) \end{aligned}$$

JAS. W. STEWART.

1725. The equation $a^b = b^a$.

Eric Goodstein proposed the problem of finding all the solutions of the equation

$$a^b = b^a.$$

The class of solutions clearly has the power of the continuum, for the equation is equivalent to $(\log a)/a = (\log b)/b$, and $(\log x)/x$ is steadily increasing as x increases from 1 to e , and steadily decreasing as x increases from e to infinity, so that $(\log x)/x$ takes every value from zero to $1/e$ for exactly two values of x , one less than, and the other greater than e .

To find the general solution, write $c = \log_b a$; then $a = b^c$ and the equation becomes

$$b^{bc} = b^{b^c},$$

whence $bc = b^c$, and so $c = b^{c-1}$, (or $b = 0$), that is

$$b = c^{1/(c-1)} \quad \text{and} \quad a = c^{c/(c-1)}.$$

Write $x = 1/(c-1)$; then

$$a = \left(1 + \frac{1}{x}\right)^{1+x}, \quad b = \left(1 + \frac{1}{x}\right)^x$$

which is the general solution.

For integral values of x , the values of a and b are rational; if x has a rational value p/q , $q \geq 2$, then both a and b are irrational, for p and $p+q$ cannot both be q^{th} powers (since from $p = r^q$ it follows that

$$(r+1)^q > r^q + qr^{q-1} \geq p+q);$$

if x is an algebraic irrational then both a and b are transcendental, and if x is transcendental, no rational value of $(1+1/x)^x$ is known.

Thus the rational solutions are given by

$$a = \left(1 + \frac{1}{n}\right)^{n+1}, \quad b = \left(1 + \frac{1}{n}\right)^n$$

for integral values of n .

There are only two integral solutions, given by $n=1$ and $n=-2$, and these are

$$a=4, b=2 \quad \text{and} \quad a=2, b=4.$$

R. L. GOODSTEIN.

1726. On Note 1644.

Whilst experimenting with Mr. Cohn's test for divisibility one or two points occurred to me which, although fairly obvious, are not explicitly brought out in the Note and so may be worth recording.

(i) We may take advantage of the fact that s need not be taken to be positive to give us the most convenient value, e.g. if $p=23$, the labour is reduced by taking $s = -7$ rather than $s=16$.

(ii) Since the only restriction on p is that it should be prime to 10, there is no advantage in applying the method only to prime divisors. E.g. to test divisibility by $13^2 (=169)$ it is simpler to take $s = -17$ than to test twice for divisibility by 13 as suggested.

(iii) The rule will work for all divisors of the forms $10k+1$, $10k-1$, $10k+3$, $10k-3$, and the (numerically) smallest corresponding values of s are k , $-k$, $-1-3k$, $-1+3k$ respectively.

(iv) The method becomes a trifle neater if we change the sign of s throughout (so that $p \mid 10s-1$, and we add sA_0 in applying the test, which means, arithmetically, an addition when s is positive and a subtraction when s is negative instead of *vice versa*).

A. R. PARGETER.

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REVIEWS.

The Case for Examinations. An account of their place in education with some proposals for their reform. By J. L. BRERETON. Pp. viii, 226. 8s. 6d. 1944. (Cambridge University Press)

This book is an important contribution to the discussion of current educational problems; its author is an authority whose experience of examinations, which few can exceed, gives weight to his opinions. It may be read as, in some ways, a counterblast to the Norwood report, though it was practically finished when that report was issued, the report itself being discussed in an added final chapter.

The book is in two parts: Part I, General and Historical; Part II, Proposals for Reform; and those readers who do not find themselves in close agreement with Part II are at any rate certain to find much to interest them in Part I.

It seems desirable in this review to give an account of the book chapter by chapter and to do so largely in the author's own words.

In Part I the first chapter, "Incentives to Learning" is vital to the whole argument, for it emphasises that "examinations fulfil a double function—they are a mobilising force in education and they provide a means of testing its results. To ignore either of these aspects is to get a distorted view of the whole subject". It is the neglect or rather the denial of the first aspect that seems to the author to invalidate many of the conclusions of the Norwood report. He maintains that examinations not only *do* but *must* influence the work in schools and that, granted reform in certain respects, they definitely *should* do so. "The success of an examination as a stimulus to effort depends on four factors: (1) a suitable reward for success, (2) a limited time, that is to say, the examination cannot be postponed indefinitely, (3) knowledge that the examination will be conducted fairly and honestly, (4) knowledge that it will be neither too hard nor too easy for the students who are to take it."

Chapter II, "Standards of Attainment". In this chapter Mr. Brereton speaks with complete authority, and his main conclusion is that "the search for an absolute standard of skill is a fruitless one". "The policy of the examining bodies in relation to School Certificate subjects is to give credit to about half the candidates." Thus "the standard of an examination adjusts itself to the standard of those taking it. This applies both to the standard of questions and to the standard of result expected".

Chapter III, "Links in Education", first explains that the true distinction between an external and an internal examination is not that the former is set by an outside examiner, but that it is taken by pupils from several schools or classes. "The cramping effect of external examinations on teaching, which has received so much attention lately, is traceable, not to the fact that students from several schools take the same examination, but to the questions and tests being set by an outside body insufficiently in touch with the teachers." "An examination is a link between two phases in the education of the students taking it. From the educational point of view we want to achieve a connected course of training in which each stage leads on to the next. When we say that an examination should be related to the future needs of the students, we are merely saying that their *training* should be related to their future needs."

Chapter IV, "Development of Examinations before 1911." Chapter V, "School Examinations, 1911 to 1942." These give a highly interesting account by one who knows the facts. More controversial matter is reached in comments on the Investigators' School Certificate Report (1932) and the

Spens Report (1938). "It is an interesting fact that the (former) report, although paying homage to the cardinal principle that the examinations should follow the curriculum and not determine it, is full of proposals for using the examination to modify the curriculum." "The slogan 'easy papers and a high standard of marking' has seemed to me to be an unhelpful one. It is quite possible to set a paper which is so easy that no degree of severity in the marking will differentiate the candidates from one another." "The Spens Report failed to disclose the real trouble, which was that the School Certificate examination did not differentiate sufficiently between the better and the average students. By making the pretence that all School Certificates should be of equal value the authorities played into the hands of 'matric'."

Part II. "Proposals for Reform." Chapter VI. "Development of Syllabus." "The general plan proposed for dealing with subject syllabuses is the setting up of a series of standing representative national councils, one for each main subject, to be charged with the coordination of existing syllabuses and their adaptation to present needs. The Subject Councils would take over from the examining bodies the responsibility of deciding what should be taught in schools." A digression on committee work in general discusses six conditions for its success, with instances showing how absence of some of these conditions has led to failure.

"The coordination of School Certificate syllabuses ought not to present great difficulty in any subject because the bodies which accept the S.C. credit for exemption purposes seem to be prepared to leave the decision as to what ground is covered by the examination in the hands of the examining bodies, and all that is needed is cooperation between the latter." It might be added that for mathematics the Jeffery committee proves this cooperation to be possible. "At the Higher Certificate level the position is complicated because of the exemptions from university Intermediate examinations and because of the College Scholarship examinations."

"Where there is a strong subject Association, this can take a leading part in securing coordination but should be content to work with the examining bodies and the four main teachers' associations."

Chapter VII. "Regional Joint Examinations." The two reforms advocated, which cannot be considered separately, are:

"(i) Schools and universities to be grouped on a regional basis, so that (a) each examining body examines only the schools in a geographical region surrounding its own university, and (b) all schools in this region take the same examination. In devising the regions the aim should be for each examining body to have a share of eight to twelve thousand of the present seventy thousand candidates."

"(2) The constitution of each examining body to be on the basis of one-quarter to one-third representatives of the university; one-third to one-half representatives of the teachers in the schools examined; one-quarter to one-third representatives of the Local Education Authorities in the region, and one or two H.M. Inspectors."

"Competition between the examining bodies has certainly played a part in maintaining their efficiency; but the consequent haphazard distribution of the schools taking any given examination is one of the main obstacles to the further development of control by the schools." The Durham and Bristol Boards "have been able to provide the small number of schools which take their examinations with many of the advantages of regional joint examinations". "The influence of the Oxford and Cambridge Joint Board in maintaining the isolation of the public schools at a time when everyone is agreed that they should be brought more closely into touch with the rest" is serious.

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The teacher representatives "should be appointed directly by the schools taking the examination". There should be "no compulsion without a share of control".

Chapter VIII. "Less Academic Subjects." Here "mathematics follows naturally after English Language because these two subjects are the servants of the other subjects of the curriculum. They both deal with the expression of ideas by symbols; their principles are essentially for application." "Perhaps the most controversial section of the subject is Geometry." "The Mathematical Association has done much through its *Report on the Teaching of Geometry*, and through discussions, to widen the scope of the subject." After a few paragraphs dealing with other subjects, the bulk of the chapter is devoted to Science and the Spens Report.

Chapter IX. "The Board of Education's Part." "Overdue improvements, made imperative by the war, will be found to have prepared the way for further advances. Thus, the complete cessation of the activities of the Secondary School Examinations Council during the war may be an indication of its unsatisfactory constitution; while the way in which examinations have been carried on in spite of the difficulties of the times bears witness to their importance in the educational system." "People who have to deal with it find the Board a most elusive body. Even its name is misleading, since, for all practical purposes, there is no 'Board'. One suspects that all decisions are taken individually or jointly by civil servants who are responsible in the last resort only to the President of the Board and so to Parliament."

Elaborate diagrams are given to show how the various bodies concerned in examinations should be coordinated, and the chapter ends with a list of seven steps (towards reform) which could be taken at once or at any time.

Chapter X. "The Norwood Report." Here the author is in opposition, especially to the whole idea of an internal examination at the School Certificate stage. His view is well expressed in the first paragraph, of which the rest of the chapter is an amplification.

"Examinations came into being a century ago as a counter to arbitrary selection and private patronage, and they have remained a bulwark of fair competition ever since. They have gradually developed as an integral part of the educational system and have come to fulfil an important role in linking schools with each other and with the rest of the world. We have traced to their causes some of the objectionable features of the present examinations, and have indicated the kind of development and reorganisation needed to remove them. We have also called attention to the obstacles in the way of this reorganisation. The Norwood Committee, instead of grappling with these obstacles, condemns the whole system of examinations and envisages the substitution of internal for external examinations, a step which I believe would allow arbitrariness, favouritism and patronage to raise their ugly heads again, and would cause a much greater disintegration of the secondary system than is yet fully realised."

C. O. T.

The New Education Bill. By H. C. DENT. Pp. 32. 9d. 1944. (University of London Press)

Mr. Dent has made an admirable and appreciative survey of the new Education Bill. This Bill has undoubtedly avoided political pressure to set up a simple and unimaginative system. Some of the variety, at least, has been preserved, which is the saving grace of educational organisation.

It was inevitable that the County Councils and County Borough Councils should be given greater responsibility. There is no other administrative instrument available. But those who are interested in the educational fate of children are inclined to reflect rather grimly that these local authorities

have had forty years to make their fortune and have produced a dead level of mediocrity and stagnation everywhere.

Teachers leaving the universities acquire "encirclement" neuroses in the course of a very few years.

An independent school under a good headmaster can acquire distinction in ten years. The local authorities seem able to prevent this in the case of their own schools.

However, the Bill gives new and necessary powers to the Board of Education which may counterbalance the reactionary quality of the local authorities.

We shall need more independent schools, and pressure will have to be brought on the local authorities to permit initiative and imagination inside their own schools.

One aid to this would be implementing rigorously the provision in the Bill for boards of governors for each school. A majority of these should be parents, who would provide a counter-irritant to the complacency and apathy of the local authorities.

The deliberate policy pursued in the last twenty-five years of closing small schools so that children must be taught in large classes will also have to be considered on its merits.

E. C. CHILDS.

Graded Arithmetic Tests. By A. WISDOM. Edited by P. B. BALLARD. Supplementary exercises to *Fundamental Arithmetic*. Pupil's books: I. Pp. 24. 6d.; II. Pp. 24. 6d.; III. Pp. 32. 7d.; IV. Pp. 48. 8d. Answers: I. Pp. 8., 6d.; II. Pp. 8. 6d.; III. Pp. 11. 7d.; IV. Pp. 16. 8d. Stiff paper backs. 1943. (University of London Press)

Each page of the Pupil's Books contains two sets of 12 sums: A. (Mental), requiring written answers only, and B. (Written), for which clear working is to be shown. The Tests are based on Ballard's *Fundamental Arithmetic Books* and run parallel to them, as indicated by reference on each page of Tests. They cover the syllabus very completely, each set being both thorough and comprehensive. The sums are concise in form, and the numbers involved are well selected. Scope for intelligence is provided by special problems and by varied ways of presenting mechanical exercises, the missing number form being freely used, e.g. $101d. = 8s. *d.$ The books provide a maximum of practice in a minimum of space, but they can hardly be called up-to-date, for it is nearly twenty years since Ballard's *F.A.B.* first appeared. No drawing or measuring is required, though there are sums based on drawings. Larger numbers are used than is now usual at the same stage, and the syllabus is wider. Nevertheless the books might be used by older pupils or good A. sets.

There are, however, faults in the wording of sums which detract from the value of the books. In this respect the standard is not that of *F.A.B.* Here are some examples from Book I:

1 A. 12. "What other pairs of numbers added together will make 10?"

See 1 B. 12.

22 B. 7. "If 8 is one factor of two thousand, what is the other?"

Note: "If", "the other". See 5 B. 9.

15 B. 11. "... would be required ... if they have ..."

17 A. 9. "How many pieces of wood $1\frac{1}{2}$ ins. long would make 1 foot if joined together?" See 18 A. 11.

21 B. 12. "Can you find six different ways ...?" See other questions not directed to the precise point, e.g. 14 B. 12.

13 B. 12. "... how much more would he then have than Jim?"

9 A. 10. "How many halfpennies is worth $7\frac{1}{2}d.$?" See 9 B. 10.

Judging by sample pages the other Books are similar, e.g.:

II. 16 B. 7. "At a concert 38 children paid 2d. each admission and 50

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children 4d. each." It means, presumably, "38 paid 2d. each for admission to a concert." R. S. W.

Navigation for Aircrews. II. By J. E. C. GLIDDON and E. C. HEDGES. Pp. vi, 161. With plotting chart. 3s. 1943. (University of London Press)

The little book completes most of the Navigation and Meteorology required for the Advanced Training syllabus of the A.T.C. There are several misleading statements which might confuse a student whose aeronautical knowledge was not limited to the subject-matter of this book. The inclusion of a large number of practical hints, and greater emphasis on the difficulties and unavoidable errors of practical navigation, would have made this book more generally useful. Many of the processes used in navigation are only approximations, and the student should be given some justification for the use of these methods, and some idea of the errors he should reasonably expect.

On p. 79 an erroneous method of laying off a distance on Mercator's Chart is given as an alternative to the usual one. On p. 84 the accuracy achieved in transferring a position line is not shown as depending on the track assumed, which is not always known accurately. Better methods have now been devised for use in such cases. On p. 86 occurs a useful illustration of the meaning of air speed, which is, however, marred by an earlier statement about air speed which is only true if the aircraft is flying straight and level.

Two pleasing features about the book are the clear distinction made between Position Errors and Corrections, and the full and lucid treatment of Turning and Acceleration Errors. A plotting chart is included for use in working the examples, and a useful summary of Meteorology is given in the last four chapters.

Details needing correction in a future edition are the method of indicating wind vectors, the symbol for Air Position, and the substitution of Rectified Air Speed for Corrected Air Speed.

K. R. I.

Aircraft Navigation. I. Theory. By H. STEWART and A. NICHOLS. **II. Practice.** By S. A. WALLING and J. C. HILL. Pp. iv, 146, with plotting chart and circular slide rule. 5s. 1943. (Cambridge University Press)

This compact book covers the A.T.C. syllabus for the new Advanced Training examination in Navigation and Meteorology, with chapters on Radius of Action and Interception in addition. It should be useful for revision purposes to cadets and prospective members of aircrews who have no easy access to the official R.A.F. manuals on the subject, and who wish to supplement the instruction they receive.

The chief advantage of the book lies in its treatment of Navigation Theory, Plotting, Instruments, Meteorology and Map Reading, and Star Recognition, in one small volume, but the inevitable result is that it suffers in some places from a lack of detail and explanatory diagrams. Apart from a few errors and omissions, the subject-matter is accurate and up-to-date, and contains new presentations of essential topics, while introducing other ideas not usually found in books of this type. For example, the section on log-keeping, though necessarily brief, introduces the reader to a subject which is of primary importance in air navigation. Purchasers of the book are supplied with a plotting chart and a circular slide rule, and a section of the book is devoted to an explanation of the latter.

One serious disadvantage is the lack of practical details, and the unsuspecting reader may come up against grave difficulties when he finds himself in the air. The examples for working are adequate in number but rather unimaginative in conception. On p. 24 an inaccurate method of measuring distances is given. The error involved is not large, but an intelligent pupil

might detect it and suppose that accuracy did not matter; in any case the correct method is just as simple to use. On p. 91 no explanation is given of how to apply variation for a long flight. In spite of these criticisms one feels that the book is well worth reading, and that it will be found useful to all those interested in navigation and its allied subjects.

K. R. I.

Illustrated Calculation for the A.T.C. By T. H. WARD HILL. Pp. 111. 3s. 1943. (Harrap)

This is not another of those books on Mathematics for the A.T.C. which a cadet picks up hopefully and puts down with a sigh—or worse! This attractive volume, like *Riders in Geometry* by the same author, deserves to be read by all intending teachers of elementary mathematics, as well as by the A.T.C. cadets for whom it is primarily written. The author has admirably carried out his avowed purpose of conveying the elements of mathematics to those who have formerly been bored or baffled by the subject. He has cut adrift from the usual school textbooks and presented even fractions, decimals and percentages in a simple and interesting manner, while drawing attention to the useful applications of these topics. Frequent use has been made of aeronautical illustrations and data, and the reader is constantly reminded of the purpose for which he is studying the book. A similar book on Basic Mathematics written in Basic English would be a welcome substitute for the usual school textbook, and none could be better qualified than Mr. Ward Hill to write it.

K. R. I.

MATHEMATICAL FILMS.

IN January 1943 the British Film Institute established at the instigation of the British Council a sub-committee to consider the possibilities of Mathematical Films. The committee at its first meeting consisted of Mr. A. E. Evans of the Association of Teachers in Technical Institutions, Mr. J. Fairgrieve, Instructor Commander H. A. McDonald, and myself as the M.A. representative. Mr. Oliver Bell, director of the B.F.I., lent his genial presence as coordinator.

Three films were shown: *Die Entstehung der Kurve* (Germany, 1938), $\ddot{x} + x = 0$ (Fairthorne and Salt, 1939), and *Mouvements Vibratoires* (France, 1936). The committee felt that the first film was too complex, and the second too surrealist, but that the French film with its happy use of both photography and diagram represented the most suitable technique for instructional purposes. It was agreed that (i) subjects such as calculus and mechanics, which involve growth and movement, would be most suitable for film treatment; (ii) the duration of a film for 14-year-olds should not exceed five minutes; (iii) a start should be made on two or three short films dealing with simple harmonic motion and its relation to wave motion, progressive and stationary.

Any member of the M.A. who is prepared to help by his presence on the sub-committee, or who has ideas for mathematical films, should write to me:

A. P. ROLLETT,

OAK LANE,

SEVENOAKS, KENT.

It might be added that the British Council proposes to circulate these films abroad as examples of British educational work.

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NEWS OF BRANCHES.

SCHOOL CERTIFICATE MATHEMATICS.

Manchester Branch. On 7th March a discussion on the new alternative Cambridge syllabus in geometry and trigonometry was opened by Miss Garner. The meeting approved the drastic reduction in the number of theorems to be proved, and felt that there was much to be said for excluding all proofs of theorems; there was general disapproval of any reduction in the number of facts to be learned and a desire to have a list of all facts that might be assumed. The theorems given were not accepted as key theorems. The fusion of trigonometry and geometry was approved, and members welcomed the correlation of mathematics with A.T.C. work, geography, physics and engineering, but disapproved of examination questions which would require a specialised knowledge of these subjects. The syllabus as a whole was considered an unsuitable preparation for the Higher School Certificate work.

Midland Branch. The new Cambridge syllabus has been discussed and its general principles approved unanimously. Criticism of the list of theorems followed the lines of those already sketched in the *Gazette*. Further criticism concerned the paragraph A (Theoretical) on p. 2: "Riders will be straightforward and will be worded so as to make clear not only what is to be proved but also what is to be assumed. Subject to any such restriction, candidates may, in proving riders, make use of all their knowledge of geometrical facts." The meaning and purpose of these sentences is far from clear, as the following considerations may show.

It may be thought that a student who has pursued a course of geometry should know certain facts and have learned to select those of them which are required for the solution of a particular problem. If this is so, it would appear unnecessary or actually undesirable to give the theorem to be used. In questions of the specimen paper only the chief theorem to be used is given, though other properties will in fact be assumed. This conflicts with the first of the two sentences quoted above. Again, some of the theorems to be assumed appear to be outside the stated syllabus, if this is really to omit much of the formal geometry.

These questions therefore arise: (a) Do those who framed the new syllabus in fact expect the candidate to have covered substantially the same course as under the older syllabus, together with the extra trigonometry and practical work, but with the burden of rigorous proof of quotable results restricted to a small number of theorems? (b) If not, is it their intention that riders should be set in which candidates will be instructed to use results previously unknown to them? This would obviously be an important new principle. (c) What is the real purpose of stating "what is to be assumed"? It has sometimes been said, mainly in connection with the reproduction of short pieces of bookwork, that candidates do not know what to assume. In a short series of theorems set out in a prescribed order the difficulty disappears. In rider-work too it disappears if candidates "may make use of all their knowledge of geometrical facts". In some of the questions of the specimen paper there is almost nothing left for the candidate to do, when he has been told what to assume.

North-eastern Branch. The first war-time meeting of this Branch, on 11th March, was addressed by Dr. J. Hargreaves on "Alternative Syllabuses in School Certificate Mathematics". Among many interesting topics discussed by the lecturer were:

Properly conducted examinations should have a stimulating effect on most

students: he deprecated the ban on the use of parallel rulers: he suggested the inclusion of the following topics: simple statistics, actuarial calculations, hire purchase and house purchase calculations, probability. He thought that the existing Geometry syllabus should be retained for the A stream pupils and that the proposed alternative syllabus (Cambridge) would benefit the B and C streams.

The meeting approved the idea of including Trigonometry with Geometry, the Trigonometry part of the syllabus, the idea of a small number of "key" theorems, the publication of a list of theorems in groups, but did not approve of a fixed sequence; formal proofs may be required of the area group but not of the similarity group, theorems may be tested by formal proofs, but the meeting was against the inclusion of plans and elevations in the curriculum unless questions were illustrative of solidity—not too technical.

It was decided to hold a meeting in the Summer term and to re-open Branch activities.

Yorkshire Branch. As reported in the February *Gazette*, this Branch has established a committee to consider the School Certificate syllabus and related topics. After much discussion on the future position of mathematics in schools, the committee felt that the syllabus for the first two years, from the age of 10+, should be the same for Grammar and Technical schools, but that it might require modification for the Modern school in the light of evidence that might become available. This would allow of transference from one school to another at the end of this time. For the following three years there should be a different syllabus for each type of school, suited to its needs.

LONDON BRANCH.

On 18th March a discussion on the "Place of Arithmetic in Education" was opened by Mr. A. J. G. May, who advocated the inclusion in the syllabus of Arithmetic of Citizenship in place of much that was of doubtful value.

During the discussion it was agreed that there must be taught a certain core of mechanical arithmetic and that this should be illustrated in as wide a field as possible. Emphasis was laid on the importance of the automatic performance of the fundamental operations. The next meeting of the Branch will take place at 2.45 p.m. on 20th May at the Polytechnic, Regent Street.

BOOKS RECEIVED FOR REVIEW.

- J. L. Brereton. *The case for examinations.* Pp. viii, 226. 8s. 6d. 1944. (Cambridge)
 H. C. Dent. *The new Education Bill.* Pp. 32. 9d. 1944. (University of London Press)
 E. W. Golding and H. G. Green. *Elementary practical mathematics. II.* 2nd edition. Pp. xii, 202. 7s. 6d. 1942. (Pitman)
 G. H. Hardy and W. W. Rogosinski. *Fourier series.* Pp. viii, 100. 8s. 6d. 1944. Cambridge Tracts, 38. (Cambridge)
 T. H. Ward Hill. *Illustrated calculations for the A.T.C.* Pp. 111. 3s. 1943. (Harrap)
 A. Wisdom. *Graded arithmetic tests.* Supplementary exercises to *Fundamental arithmetic.* Edited by P. B. Ballard. Pupil's books: I. Pp. 24. 6d.; II. Pp. 24. 6d.; III. Pp. 32. 7d.; IV. Pp. 48. 8d. Answers: I. Pp. 8. 6d.; II. Pp. 8. 6d.; III. Pp. 11. 7d.; IV. Pp. 16. 8d. 1943. (University of London Press)

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